New developments in Realistic Mathematics Education: the *Beyond Flatland* project

Marja van den Heuvel-Panhuizen & Michiel Veldhuis
Reform in the Netherlands started in ~1968

Mechanistic Mathematics Education
Focus on procedural skills

Realistic Mathematics Education
Focus on conceptual understanding

2018
### 1. Math Problems

<table>
<thead>
<tr>
<th>1.</th>
<th>20/1480 \ 30/2190 \ 40/2160 \ 50/3400 \ 60/4860 \ 20/1640 \ 90/4680 \ 70/2170 \ 80/6560 \ 70/3710 \ 80/5120 \ 60/1620</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.</td>
<td>430 \ 19 \ 321 \ 28 \ 212 \ 37 \ 203 \ 46 \ 142 \ 55</td>
</tr>
<tr>
<td>3.</td>
<td>1458 \ 2057 \ 143 \ 17 \ 567 \ 3296 \ 25 \ 4647 \ 2048 \ 372 \ 5788 \ 2348 \ 1356 \ 2973 \ 738 \ 2367 \ 4 \ 815</td>
</tr>
<tr>
<td>4.</td>
<td>7100 \ 3675 \ 8000 \ 4783 \ 6042 \ 5291 \ 3810 \ 2170 \ 7002 \ 6999</td>
</tr>
</tbody>
</table>

### 5. Word Problem

Bob, Wim and Koos have $f 6$ together. 
Bob has $f 0.95$ Koos has $f 3.15$ Wim has $f ...$

### 6. Tables

| 1 jaar = ... mnd. | 26 mnd. = ... jaar + ... mnd. |
| 3 jaar = ... mnd. | 39 mnd. = ... jaar + ... mnd. |
| 6 jaar = ... mnd. | 63 mnd. = ... jaar + ... mnd. |
| = ... mnd. | = ... mnd. |
**Straightforward task**

In class there are 28 children. Each child will get a holder with six pencils. How many pencils need the teachers for this?

![Image of pencils]

**Gray area tasks**

Pay the exact amount. Try it in at least five ways.

![Image of a vase with 3675 and a flower]

**Puzzle-like task**

\[
\begin{align*}
20 - \phantom{00} &= \ldots \\
20 + \phantom{00} &= \ldots \\
20 \times \phantom{00} &= \ldots \\
\text{Together 160}
\end{align*}
\]
Non-Routine Problem Solving Tasks in Primary School Mathematics Textbooks – A Needle in a Haystack

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ABSTRACT: In this paper, we report on a study in which we investigated the nature of numerical problem solving tasks as presented in primary school mathematics textbooks in the Netherlands. Although several factors influence what mathematics teachers teach children, there is much evidence that the curriculum and the textbooks are important determinants of what children are taught and what they learn. Contradicting results for performances of Dutch fourth graders on a test on mathematical reasoning for this textbook analysis study.
Results textbook analysis

![Bar chart showing the percentage of puzzle-like tasks and gray-area tasks across different textbook series.]

- **De Wereld in Getallen**: 13% puzzle-like tasks, 12% gray-area tasks.
- **Talrijk**: 12% puzzle-like tasks, 13% gray-area tasks.
- **Pluspunt**: 7% puzzle-like tasks, 8% gray-area tasks.
- **Rekenrijk**: 5% puzzle-like tasks, 8% gray-area tasks.
- **Wis en Reken**: 4% puzzle-like tasks, 6% gray-area tasks.
- **Alles Telt**: 2% puzzle-like tasks, 5% gray-area tasks.

% of units

0 2 4 6 8 10 12 14 16 18 20

- **Puzzle-like tasks**
- **Gray-area tasks**
The new **goals** of Wiskobas were

- generalising
- proving
- **mathematising**
- schematising
- symbolising
- using models

and they covered the **subject areas**

- arithmetic
- measurement
- geometry
- probability and statistics
- relations and functions
- language and logic
Block schemes to solve equations

- Think of a number.
- Add 10: \(+10\)
- Multiply by 2: \(\times 2\)
- Subtract the number you started with.
- Subtract again the number you started with.
- Write down the result.

DONE

Final result: \(20\)
Time-distance graphs with a story

Arrival: train
Bus to city center
Visit to city hall
Walk
Destination

1975
Reasoning about probability

**Kijk op kans**
Janssen & Goffree, 1972/1973
4-year project: 2015-2019

Senior staff

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Theoretically enhanced by

- Embodiment theory
- Representational re-description theory
- Variation theory
- Our sensori-motor system has an important role in developing conceptual understanding
- The same neural substrate used in imagining is used in understanding (Gallese & Lakoff, 2005)

- Embodiment theory

- Representational re-description theory

- Variation theory
- Our sensori-motor system has an important role in developing conceptual understanding.
- The same neural substrate used in imagining is used in understanding.
  (Gallese & Lakoff, 2005)

“human ideas . . . are organized in vast (mostly unconscious) conceptual systems grounded in physical, lived reality”
  (Núñez, Edwards, & Matos, 1999, p. 50)

- Embodiment theory

- Representational re-description theory

- Variation theory
The RR theory describes the development of representations, which can bring students to higher levels of thinking. The initial implicit, embodied knowledge, is in a next step re-described in verbal or other types of symbolic representations and, as such, becomes available for explicit verbal-symbolic reasoning and explicit hypothesis-led experimentation.

(Karmiloff-Smith, 1992)

- Embodiment theory
- Representational re-description theory
- Variation theory
A necessary condition for learning is the possibility to experience variation and distinguish between what changes and what remains invariant. (Marton & Booth, 1997; Marton & Pang, 2013)

Being able to discover structure and to identify patterns is considered the essence of mathematics (Watson & Mason, 2006)

Therefore, variation theory is considered a powerful design principle for mathematics education (e.g. Sun, 2011; Li, Peng & Song, 2011)
Beyond Flatland in primary school mathematics education

How to teach EARLY ALGEBRA
Key components of the Flatland teaching sequence for EARLY ALGEBRA

Focus on: Algebraic reasoning with linear equations

More specifically: Reasoning with, and about, unknowns using algebraic strategies

Context: Working with a hanging mobile

Embodiment: Experience of balance - equality
Structure of the Flatland teaching sequence for EARLY ALGEBRA

1. Reactivating the concept of equality; informally use algebraic strategies
2. Eliciting algebraic strategies of restructuring and isolation
3. Eliciting algebraic strategies of isolation and substitution
4. Applying algebraic strategies in a different context: tug-of-war
5. Applying algebraic strategies in a different context to find values
6. Applying algebraic strategies in a formal context to find values

- One equation
  - Physical hanging mobile(s)
  - Relations between unknowns
- Informal notations
- Towards more formal notations
- Values of unknowns
- System of equations
  - Other contexts
- Informal notations
What can you do to keep the hanging mobile straight?
What can you do to keep the hanging mobile straight?
What can you do to keep the hanging mobile straight?

Use the algebraic strategies of

- **Restructuring** by
  - Changing sides
  - Changing order of bags on the same side

- **Isolation** by
  - Taking away similar bags on both sides
  - Taking away different bags on both sides

- **Substitution** by
  - Replacing bags by bags of another color
Eliciting algebraic strategies: restructuring & isolation

Can you make the hanging mobile clearer and discover the secret rule?

<table>
<thead>
<tr>
<th>Make clearer</th>
<th>Draw as few bags as possible</th>
<th>What is the secret rule?</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Diagram" /></td>
<td><img src="image2" alt="Diagram" /></td>
<td><img src="image3" alt="Diagram" /></td>
</tr>
</tbody>
</table>

Remove the same number until that isn’t possible anymore and that’s it!

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Combine the information from the two given hanging mobiles to find the other relationship.

Drawn with green and white the right amount of bags. Draw as few as possible!
Combine the information from the two given hanging mobiles to find the other relationship.
Applying algebraic strategies in a different context

Which hanging mobile fits the tug of war situation?

Equally strong

Equally strong

Equally strong

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Applying algebraic strategies in a different context

Equally strong

Who will win?

[Images of animals and equations]

\[ \text{Equation: } x + y + z + u = 7 + p + v + 3 \]
Applying algebraic strategies in a different context

\[ a = \text{aardbei} \]
\[ b = \text{banaan} \]

\[ \text{afb} \ 2 \]

\[ p = \text{per} \]
Applying algebraic strategies in a different context

1. Klopt dit? JA / NEE
   Hoe weet je dit?

   bij afb 1 doe je de peer voor
   1 a en 1 b
   je streelt links
   en recht aardbei
door dan heb je
   nog 2 a en 5 b
   dus dat gedeelt
   door 2 =
   1 a
   goe
   1 a en 3 b

2. Klopt dit? JA / NEE
   Hoe weet je dit?

   1 b + 1 a = p
   1 a = 2½ b
   p = 1 a + b
   1 a = 2½ b
   2½ + 1 b = 3½
   en geen u

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Applying algebraic strategies to find values - informal

First write this is-equal-to task differently, then find out what needs to be filled in:

\[ 3 \text{ \ Euro} \text{ ] + 2 \text{ \ Euro} = \text{ \ Euro} 22 \]

\[ 4 \text{ \ Euro} = 1 \text{ \ Euro} \]

First write this is-equal-to task differently, then find out what needs to be filled in:

\[ 2P + 4S = 17 \]

\[ 1P + 1S = 5 \]

\[ P = \text{ \ Euro} \text{ } \]

\[ S = \text{ \ Euro} \text{ } \]

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Applying algebraic strategies to find values - formal

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\[ \begin{align*}
M + 3L & = 25 \\
2M & = 4L \\
M + 5L + L & = 25
\end{align*} \]

Substitution

Isolation

Restructuring

Substitution

\[ M = 10, \quad L = 5 \]
Key components of the Flatland teaching sequence for EARLY ALGEBRA

Focus on: Algebraic reasoning with linear equations

More specifically: Reasoning with, and about, unknowns using algebraic strategies

Context: Working with a hanging mobile

Embodiment: Experience of balance - equality
How to teach DYNAMIC DATA MODELING
Key components of the Flatland teaching sequence for DYNAMIC DATA MODELING

Focus on: *Reasoning about graphical representations of change*

Specifically: *Reasoning about, and interpreting, time-distance-graphs*

Context: *Moving in front of a motion sensor*

Embodiment: *Experience of moving through space – graph (covariation)*
Structure of the Flatland teaching sequence for DYNAMIC DATA MODELING

1. Explore motion: reflecting and representing
2. From discrete to continuous representations of change
3. Continuous graphs of ‘distance to’ (1)
4. Continuous graphs of ‘distance to’ (2)
5. Scaling on the graphs’ axes
6. Multiple movements and their graphical representation

- Informal graphs
- Continuous graphs
- Discrete graphs
- One graph/situation
- Separate time points
- Motion sensor
- Time as continuous variable
- Multiple graphs/situation
Represent your trip from home to school
Explore motion: reflecting and representing

Represent your trip from home to school

[Diagram of a route from home to school with times marked at various points.]
Who arrives first?

1. A person walks normally towards the middle and then slowly towards the end.

2. A person walks fast towards the middle, stands still for two seconds, and walks then normally towards the end.
Who arrives first?
Intruder problem

HET INBREKERSPROBLEM

Naam: ____________________________

In een geheime laboratorium ergens op de wereld worden nieuwe plantsoorten ontwikkeld. Zaaikapje van deze planten zijn extrem steenzaam en heel erg waardevol.

Maar... er is iets wels te beurt! Midden in de nacht heeft een inbreker zaadjes van een van deze plantsoorten meegenomen! Vanwege de unieke kenmerken van elke plant is het belangrijk dat wordt uitgezocht van welke plant de inbreker zaadjes heeft gestolen. Kun je de politie helpen dit probleem op te lossen?

Planten:

Lab 1
Naam: *Echinops Multiflora Deformis*
[Langstelige Kogeldisvel]

Lab 2
Naam: *Mirabilis Jovis Apertus*
[Open Zevendood]

Lab 3
Naam: *Aconitum Vulparia Magna*
[Grote Gele Mannskap]

Lab 4
Naam: *Elytrigia Serpyllum*
[Wilde Waterpest]

Overzicht grafieken:

Grafieken [DEEL 1]

Gedurende de nacht zijn een aantal grafieken gemaakt. De inbreker was van tijdop 01:30 uur en 01:43 uur in het gebouw.

Je vindt hier zeven grafieken. Er is een grafiek voor bewegingssensor 1 en een grafiek voor bewegingssensor 2. Vier grafieken geven de temperatuur in de verschillende ruimten weer.
Intruder problem

Gedurende de nacht zijn een aantal grafieken gemaakt. De inbreker was van tijdstip 01:30 uur en 01:43 uur in het gebouw. Je vindt hier zeven grafieken. Er is een grafiek voor bewegingssensor 1 en een grafiek voor bewegingssensor 2. Vijf grafieken geven de temperatuur in de verschillende ruimten weer.

Overzicht grafieken:
Intruder problem

Question:
From which room(s) did the burglar steal some seeds?

Available information:
- Floor plan
- Time the burglar is in the building
- 2 motion graphs
- 5 Temperature graphs
Walking continuous graphs of distance to

Motion sensor

Continuous graphs (1)
Walking continuous graphs of distance to
Walking continuous graphs with changing speeds
Scaling on the graph’s axes
Scaling on the graph’s axes

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Multiple movements and their graphical representation
Key components of the Flatland teaching sequence for DYNAMIC DATA MODELING

Focus on: Reasoning about graphical representations of change

Specifically: Reasoning about, and interpreting, time-distance-graohs

Context: Moving in front of a motion sensor

Embodiment: Experience of moving through space – graph (covariation)

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How to teach EARLY PROBABILITY?

**Common approach**

- Doing experiments
- Seeing what comes out
- Explaining the results
- Exploring the sample size

**Our approach**

- Exploring the sample space
- Predicting what comes out
- Doing one experiment
- Doing many experiments
Four key components of the Flatland teaching sequence for EARLY PROBABILITY

1. Using sample space as a starting point for probabilistic reasoning

2. Three guiding questions
   - What can you get?
   - What will you get?
   - What did you get?
Four key components of the Flatland teaching sequence for EARLY PROBABILITY

1. Using sample space as a starting point for probabilistic reasoning

2. Three guiding questions
   - What can you get?
   - What will you get?
   - What did you get?

3. Supporting perspective switches between
   - Unpredictability <-> Predictability
   - Theoretical probability <-> Empirical probability
   - Elementary results <-> Types of results

4. Experiments with physical chance generators and computer simulations
Structure of the Flatland teaching sequence for EARLY PROBABILITY

1. Exploring basic concept of probability
   - 1 chance generator

2. Exploring throwing with one die: one throw, many throws
   - 1 chance generator
   - one experiment
   - and
   - many experiments

3. Exploring two coins: elementary results and types of results
   - 2 chance generators

4. Exploring two dice: elementary results and types of results
   - 4 chance generators

5. Exploring the marble scooper
   - 4 chance generators

6. Discovering similarities between different chance situations
### Sorting events

<table>
<thead>
<tr>
<th>Certain</th>
<th>Maybe</th>
<th>Certainly Not</th>
</tr>
</thead>
<tbody>
<tr>
<td>This will <em>certainly</em> happen</td>
<td>This will <em>maybe</em> happen</td>
<td>This will <em>certainly not</em> happen</td>
</tr>
</tbody>
</table>

- The train from Utrecht to Amsterdam has tomorrow a delay of an hour
- If you always water a plant it will stay alive
- If you never water a plant it will die
- Taking a blue marble from a bag containing nine blue and one red marble
- Next week everything is for free in the supermarket
- If you enter a lottery you win a prize
- If you enter a lottery you win a prize
Exploring basic concept of probability

What can you throw? What will you throw?

**Regular die**

1. What numbers can you throw?
   \{1,2,3,4,5,6\} → Theoretical probability (sample space)

2. What number will you throw? → Notion of chance

**Adapted die**

3. What numbers can you throw?
   \{1,2,3,4,5,5\} → Theoretical probability (sample space)

4. What number will you throw? → Notion of chance

5. Think of a die with which it would be easier for you to predict what you will throw? → Manipulating the theoretical probability
Exploring throwing with one die: one throw, many throws

Predicting, throwing, and looking what you get

Regular die

1 x 6 x 30 x 1000 x
Exploring throwing with one die: one throw, many throws

Predicting, throwing, and looking what you get

Regular die 1000 x

Adapted die 1000 x

What result will you get?

→ Connecting theoretical probability to empirical probability

Look and predict

What kind of die is used?

→ Connecting empirical probability to theoretical probability
Exploring two coins: elementary results and types of results

 Ø Tossing two coins once

What can you get?

<table>
<thead>
<tr>
<th>1st coin</th>
<th>2nd coin</th>
<th>3 types of results</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>H</td>
<td>2xH</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>2xT</td>
</tr>
<tr>
<td>H</td>
<td>T</td>
<td>H+T</td>
</tr>
</tbody>
</table>

Tim chooses 2xH  Lisa chooses H+T  Richard chooses 2xT

The coins are tossed 100 times. What do you think, who will win?
Exploring two coins: elementary results and types of results

➢ Tossing two coins many times, trying it out

Tossing and stacking pieces of wood

Simulating on the computer

1000 tosses

<table>
<thead>
<tr>
<th>2xH</th>
<th>H+T</th>
<th>2xT</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Exploring two dice: elementary results and types of results

- **Throwing two dice once**
  
  What results can you get?
  
  How many combinations are possible?
  
  What sums can you get?

- **Throwing two dice many times and predicting the winning sum number**

![Combination tower](image)
Exploring the marble scooper

Scooping once

What can you scoope?
→ All elementary results

What numbers of yellow marbles can you scope?
→ All results of a particular type

Scooping many times

Gambling with the marble scooper.
What number of yellow marbles will be the winning number?

Combination tower

idea from Dor Abrahamson
Which of these situations match the above?

- Throwing a 5 or a 6 with a die.
  - YES/NO
  - Why?

- Throwing two heads with two coins.
  - YES/NO
  - Why?

- Getting red when turning the spinner.
  - YES/NO
  - Why?