Algebra: A Tool for Solving Problems?

Henk van der Kooij, Freudenthal Institute, Utrecht University
e-mail henk@fi.uu.nl

Introduction

Algebra is often seen as a powerful tool both for the modeling of problem situations and for manipulating the symbolic representation of the problem situation. These two different aspects of algebra (modeling and solving) are part of a broad discussion about algebra. In general, the learning of algebra is seen from four different perspectives (Bednarz, Kieran, Lee, 1996): the generalization perspective (generalising numerical sequences and geometrical patterns), the problem-solving perspective (often referred to as the algebraic modeling of 'word-problems'), the modeling perspective (mutual interchangebility between graphs, tables, symbolic expressions and verbal descriptions) and the functional perspective (variables and functions).
On the one hand mathematical professionals agree on the power of algebra as a tool for solving problems, but on the other hand it is well-known that a vast majority of students has a lot of problems in learning to master the symbolic, formal language of algebra with all the strict rules and conventions. So, the learning of algebra is a problem in itself too.
In this paper, some results wil be discussed of reform projects in The Netherlands and in the USA in the field of early algebra in which these problems are addressed.

Algebra or Common Sense?

The beginning of Algebra I found far more difficult, perhaps as a result of bad teaching. I was made to learn by heart:
The square of the sum of two numbers is equal to the sum of the squares increased by twice their product.
I had not the vaguest idea what this meant, and when I could not remember the words, my tutor threw the book at my head, which did not stimulate my intellect in any way.

The Autobiography of Bertrand Russell 1, p. 36

This quotation from Russell's autobiography illustrates the way in which algebra is taught and learned for a very long time. More than one hundred years after these bad experiences of Russell, this kind of exercising routine skills can still be found in Dutch textbooks, like the examples for grade 7-8 of 1985 that are shown in figure 1:

figure 1: Expanding (left) and factoring (right) algebraic expressions, both in the same chapter

The algorithmic routines of algebra are learned by heart, without understanding the meaning. But in that time (1985) there are other Dutch textbooks that also address the meaningfulness of algebra by offering contextual problems to students to be solved. Two examples with possible solving strategies based on common sense reasoning are presented here.
The first one only shows a picture at the very start of a chapter (title of the chapter: Using letters) about solving systems of linear equations, see figure 2. Without words, the picture poses a question. It is an advertisement of a petshop, in which the prices for two different combinations of turtles ('schildpadden') and goldfish ('goudvissen') are shown. Of course the intriguing question is: find the price for one turtle and for

one goldfish.

figure 2. system of equations in a picture

Just by looking at the picture, a common sense solution is easily found: the price of 1 turtle and 1 goldfish together is *f* 7. Subtract this price twice from the first combination and what remains is the price of a turtle: *f* 5. And that leaves *f* 2 for a goldfish.
In the textbook, the picture only serves as a nice illustration, because the whole chapter does not address this common sense strategy at all. The students are told, through examples, that one should solve problems like this one in the following way.
First translate the story in algebraic language and then use the algorithm to find the answer:

$$\begin{cases} 2g + 3t = 19 \\ 3g + 4t = 26 \end{cases} \begin{matrix} \times 4 \\ \times 3 \end{matrix} \rightarrow \begin{cases} 8g + 12t = 76 \\ 9g + 12t = 78 \end{cases} \rightarrow \begin{cases} 8g + 12t = 76 \\ g = 2 \end{cases} \rightarrow \begin{cases} t = 5 \\ g = 2 \end{cases}$$

The main argument used to defend this algorihmic way of solving the problem of systems of equations is that this symbolic way is more general than the common sense one. That is not true and we will come back to it later.
The second example is a situated problem that is presented here in figure 2 in an abstract format.
A triangle is given, with numbers along the edges. Every edge-number is the sum of the numbers at the two vertices, adjacent to this edge.
The question is: find the numbers at the three vertices.
This problem is presented to students on a right side page of the book and they are asked to solve the problem. On the next page, the textbook presents the algebraic way of solving it.

figure 3. the triangle problem (in Word-document "pictures")

First three strategies of students are presented and after that the algebra strategy of the book.
Student strategy 1: Add the three given numbers (63+57+100 = 220), take half of the sum (110) and subtract 100. The remainder (10) is the number on top. Similar: 110–57= 53 is the left-bottom number and 110–63 = 47 is the right-bottom number.
Student strategy 2: Start with 0 in the left-down corner. Then the number on top has to be 63, and that leaves the value –6 in the right-down corner. The numbers 0 and –6 add up to –6, but they should add up to 100. So, start again with half the difference: put 53 in the left-down corner and repeat the steps. The two other numbers are 10 and 47.
Student strategy 3: The two numbers in the bottom line have sum 100, while the left one is 6 more than the right one (because of the numbers 63 and 57). So, the left number is 50+3 and the right one is 50-3. That leaves 10 for the number on top.

The strategy of the book is (compare it to student strategies 2 and 3!):
Start with the number x in the left-down corner. Then the one on top should be $6 - ? x$ and the one in the right-down corner will be $57 - (6 - x) = x + 6$. Now we know that $x + (x + 6) = 100$, so $x = 47$ and so on.
The book strategy is, from the point of view of formal mathematics, an elegant one, because it uses only one unknown, where mostly this problem is solved by mathematical professionals using a system of three equations with three unknowns and one of the standard routines to solve such a system (through linear combination or through substitution):

$$
\begin{cases}
a + b & = 63 \\
b + c & = 57 \\
a \quad + c & = 100
\end{cases}
$$

It is not very hard to imagine that students who solved the problem without any algebraic manipulation, are surprised when a teacher or a textbook presents the algebraic formal manipulation as a powerful tool for solving this kind of problems. Apart from the difficulty of manipulating an expression like $57 - (6 - x)$, all three student strategies are much more (common)sense-making for students than the use of algebra.
What is true for these two given examples, holds for most situations where algebra is used to solve (systems of) linear equations (Van der Kooij, 1992,1999).
So, the learning of algebra can hardly be motivated by the idea that it is needed to solve contextual problems. When students solve problems in a contextual situation, they use the characteristics of the context to find a solution. Many times their strategies are a mix of common sense reasoning and calculations within the context of the problem and the use of just some (basic) algebraic work (Van der Kooij, 2001; Hoyles&Noss, 1996). Hoyles and Noss refer to this phenomenon as *situated abstraction* and students use *contextual anchors* when working on situational problems.
These considerations, based on experiences in the old program on algebra led to a new view on the learning of early algebra. In the next paragraphs, the outlines of the reform programs in The Netherlands and the USA on algebra and examples from textbooks and student work will be presented.

Algebra: Why and What?

The two reform projects are carried out between 1988 and 1998 and are different in some aspects. The Dutch project was carried out between 1988 and 1992 for grades 7-10 of secondary education. The American project was carried out between 1992 and 1998 and was meant for American middle school, grades 5-8. In many ways, the American project was a follow-up of the Dutch project: good experiences of the Dutch project were a base for refinements in the American project and bad experiences were given a second chance in the USA. A major difference between the two projects is the final aim of the program. In The Netherlands, a majority of students finish the program of algebra in grade 10, while in the USA the specific programs for algebra are part of the High School curriculum of grades 9-12.
Algebra in the reform program of The Netherlands is merely focusing on 'the study of relations between quantities, using descriptions (natural language), tables, graphs and formulas, with extensive attention for the role of variables
Some explicit choices were made:
- the focus is on the use of algebra and not on the design of algebra

Algebra is used as a language to describe relationships between quantities (the functional perspective) and for describing numeric or geometrical patterns (the generalization perspective). In both cases students are often dealing with problem situations in which the expression makes sense. Consequently:
- interpretation of an algebraic expression is more important than manipulation

Instead of mainly exercising manipulations of formal algebraic language, like in the traditional program (see figure 1), the emphasis on interpretation stimulates understanding and insight more than routine skills. Furthermore, when algebra is used to describe patterns, well-chosen problems will result in different expressions that all describe the same phenomenon (an example will be discussed later). This kind of problems provoke very useful discussions about manipulation of algebraic expressions.
- stepwise formalization (from descriptions in natural language, via so-called word formulas to the use

of letters)

This choice is based in the philosophy of Realistic Mathematics Education (see the contributions of Gravemeijer and Van den Heuvel-Panhuizen).

- starting from common sense reasoning leading in a natural (guided) way to abstract formulas and equations

As stated before, it is hard to motivate the use of algebra from simple problems that can easily be solved with just using common sense reasoning. One of the big challenges of the project in the USA was to create activities that challenged students to increasingly make use of more formal tools, based on common sense strategies.

- investigating algebraic formulas as objects (no reference to a contextual situation)

This choice is different from the other ones. In fact, activities like these focus on the understanding of formal expressions and on knowing ways to manipulate them. One important aspect of this choice has to do with so-called algebraic expertise. Expertise is more than mastering a certain amount of algebraic algorithmic routines. An expert is aware of the structure of a specific expression. In a research Wenger (1987) found that College students had a hard time to find a solution for the problem:

Solve v from $v \cdot \text{v}u = 1 + 2v \cdot \text{v}(1+u)$

Looking at the structure of the formula, one sees a 'simple' linear equation that can be solved by isolating v. Most students did not look in this way, but just tried routine based procedures they learned at school, like

- there is a square root so you have to square the equation
- I need to know v, so divide by vu
- there should be a zero on the right side, so move everything to the left side

Routine based solving procedures are of course useful, but the example shows that blindly applying a standard procedure is not a successful strategy. Dividing by vu makes the equation more complicated. When you first look at the equation, and analyse the structure of the equation, you may recognize a more appropriate way to solve the problem. Reasoning about the equation, and looking at it 'from a distance' is a strategy used by experts and therefore is a critical feature of algebraic expertise (Gravemeijer, 1990). When students are stimulated to use informal strategies to analyse and solve problems, and by generalizing the informal strategies to standard procedures for solving algebraic problems, they get a better understanding of a standard algorithm. This should prevent them from using inappropriate routine based procedures, because the stage of analysing -- looking globally at the problem -- becomes an essential part of the solving procedure.

Algebra in the Mathematics in Context Project

The purpose of the *Mathematics in Context* (MiC) project was to create a comprehensive mathematics curriculum for grades 5 through 8 that reflects the content and pedagogy suggested by the NCTM *Curriculum and Evaluation Standards for School Mathematics* (1989). The materials that have been developed are based on the beliefs that mathematics should make sense to students and that mathematics is used to make sense of the world around us. The philosophy is based on the theory of Realistic Mathematics Education, and on the belief that mathematics is fallible, changing, and, like any other body of knowledge, the product of human inventiveness. In the MiC project, the content is organized into four combined content strands: number, algebra, geometry, and statistics & probability. Forty-two instructional units spread out over the four strands have been developed.

The Dutch experiences in research and development in mathematics education of the past decades, the philosophy of RME, and the *NCTM Standards* form also the base for the approach towards algebra in the MiC curriculum. In this curriculum, algebra is characterized as:

"... the study of relationships between variables (the study of joint variation). Students learn how to

describe these relationships with a variety of representations, and will be able to connect the

representations. The algebra is used to solve problems, and students learn how to use algebra

in an appropriate manner. The latter includes making intelligent choices about what algebraic

representation (if any) to use in solving a problem. Algebra (its structure and symbols) is not

a goal on itself. *Algebra is a tool to solve problems*. The problems that arise from the real world, are often presented in contexts, they are realistic problems in the way described earlier."

This vision has been worked out in about 13 instructional units that focus on different aspects of algebra. The units in the algebra-strand of the curriculum are strongly interwoven, and they are organized around three themes.

The theme *Expressions & Formulas* involves recognizing, discovering, and then describing both geometrical and numerical patterns and regularities. Key concepts required to reason about patterns and regularities in algebra are: variables (as they play different roles), formal expressions, equations, and inequalities.

The second theme *Equations & Inequalities* is about the limitation on the values that a variable can take. Limitations or restrictions imposed by the problem situation lead to equations and inequalities that represent the problem to be solved. The MiC algebra mainly deals with (sets of) linear equations and inequalities. Students learn to interpret and validate the solution in light of the context of the mathematical equation or inequality, as well as in the context of the problem situation.

The theme *Graphs* deals with dynamic processes. A dynamic process focuses on change, especially change over time. This theme includes recognizing, describing, and using functional relationship which are represented by direct formulas or recursive formulas.

Ample time is given to develop conceptual understanding. A variety of informal and pre-formal methods is used to solve problems, and students are stimulated to use their own strategies to make sense of the problems, and then solving them. Algebraic thinking is more important than algebraic manipulating. Not until late grade 7 and grade 8, the algebra is somewhat formalized. In these last years is harvested what has been sowed earlier. Students are supported to progress from the informal level to the more sophisticated formal level during the four years of their middle school career.

Each of the three themes of the algebra strand in the MiC curriculum will be described in more detail, and in the next paragraph some examples of student materials will illustrate school practice. However, first a 'map' of the units in the algebra-curriculum is presented to provide an overview of the algebra-strand (figure 4).

This map shows how the units are organized in the three themes over the four years. The units *Dry & Wet Numbers* (dealing with positive and negative numbers), *Operations* (operating on numbers, and working with the coordinate system), and *Powers of Ten* (exponents, powers, and scientific notation) do not really fit in one of the three themes. They focus on aspects of number (another strand), and they develop concepts that are revisited and used in the units in the three themes of algebra.

The vertical order of the units indicates the chronological place of the units, and the lines in the map show the connections between the units. These are only connections dealing with prerequisite knowledge. E.g.,

Building Formulas builds on what students have done in *Expressions & Formulas* and *Operations*. As can be seen, the unit *Graphing Equations* in grade 8 is a critical unit. Much of the earlier developed algebra is tied together in this unit, and the concept of the linear relation is formalized.

The map is a helpful tool for users of the curriculum to see how the long term development of the algebra takes place in the MiC curriculum. For curriculum developers and researchers, the map can be used to illustrate features of the MiC curriculum like progressive formalization, and working at different levels of mathematical sophistication.
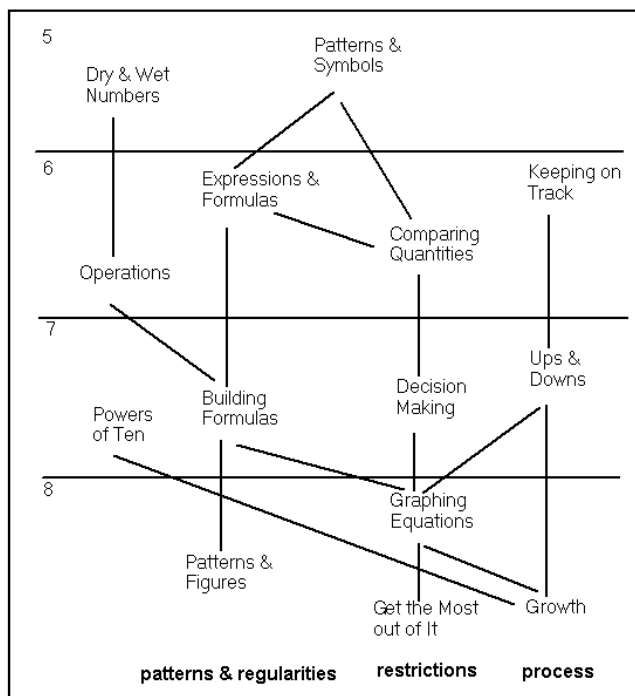

figure 4: a map of the algebra units in MiC

There is also a map for the complete MiC curriculum, and one should realize that, since MiC is a connected curriculum, algebra topics are also used (and developed) in units in other strands of the curriculum.

Expressions and Formulas (as representations of Patterns and Regularities)

The units organized around this theme focus on describing problem situations using expressions and formulas. In grade 5, regularities and patterns are observed and described. A symbol language for algebraic operations is developed. Grade 6 starts with an exploration of pre-formal types of mathematical expressions and formulas (arithmetic expressions, instructions for calculations, conventional rules, and algebraic formulas). First, chains of operations are described with the use of arrow language:

This represents in a visual way the operations carried out on a single input number to arrive at a output number, here: $(27 + 3) \times 5$. The number 27 is the only input; adding 3 and multiplying by 5 are fixed operations.

Next, this arrow language is generalized to informal word formulas which are connected to tables and charts. Possibilities to extend or to shorten arrow strings are explored, and students become aware of the fact that the order in which operations such as addition and multiplication are executed is not arbitrary.

Arithmetic trees are used to deal with order of operations that cannot be described with arrow language (like working with several input numbers or different appearances of one input number in one expression, like in $x^2 + x$).

An example of a arithmetic tree, visually representing the calculation $(27 + 3) \times 5$ is shown here.

Although the algebraic expression is exactly the same as the one with the arrows, the role of the numbers is different. There are three input numbers now and the numbers 3 and 5 are not tied to the operation.

Both informal 'visual languages' are connected to simple algebraic formulas.

In grade 7, students make their own formulas to describe more complex situations and calculations. Different representations (tables, direct and recursive formulas) are used to describe patterns and regularities. One way to describe more complex calculations is to "stack simple formulas" in a chronological order or to connect arithmetic trees. The operations are extended to squares and square roots. Furthermore, algebraic formulas and conventional rules are interpreted and tested. This includes substituting numbers, reversing formulas, and recognizing equivalent formulas.

In grade 8, the creation of formulas is extended to deriving formulas from tables with numerical data based on the discovery of patterns. Special attention is paid to linear and quadratic functions and to recursion. The context is not from real life any more, it is formal mathematics itself.

Equations and Inequalities (as representations of Restrictions)

This theme starts in grade 6 with the unit *Comparing Quantities*. Equation-like problems are represented in pictures and stories. Finding "natural" strategies (student's own invented strategies) for solving equations is encouraged. Linear equations or sets of linear equations with more than one variable are used. The problems are solved in a situation specific manner. No formal algorithms are introduced as the focus is on pre-formal methods such as trial and improve, combining and reasoning, exchanging unknowns, and using a notebook to record the manipulations with the combinations of items.

Only at the end of grade 6, a more formal notation is introduced, and discovered. By creating complex situations in which there is a need for more efficient and sophisticated strategies to solve the problems, students reinvent the algebraic language of writing (and solving) equations with letters for the unknowns. The scheme of figure 5 illustrates the variety of strategies developed in *Comparing Quantities*.

The scheme (van Reeuwijk, 1995) illustrates four different strategies -- guess & check, reasoning (or exchange or substitution), combination chart, notebook -- that are developed in the unit. The formalization

is only partly addressed in the unit *Comparing Quantities*; the concepts variable (or better unknown) and equation are further formalized and used in the units *Decision Making* and *Get the Most Out of It*. *figure 5: progressive formalization*

In grade 7, linear inequalities (boundary problems with two variables) presented in realistic situations are modeled with visual mathematical models. In this way, linear programming is foreshadowed. Constraints are represented by regions. A more formal notation with word variables and letter variables is used in the description of the problem conditions.

In grade 8, the pre-formal methods are used as a starting point for a more systematic approach to solving sets of linear equations, often connected to graphical representations. Attention is particularly paid to the graph of a linear equation and to the connection between graph and formula. Students learn to solve linear equations with one variable and sets of two linear equations with two variables. The sub-strand culminates in grade 8 with the unit *Get the Most Out of It*. This unit focuses on a rather systematic approach to linear programming.

Graphs (as representations of Processes)

This theme starts with line graphs in grade 6 where students construct and interpret graphs. This experience helps them understand the concept of a graph. Different representations are related to each other and translated into each other. All possible transitions between situation, table, and graph are emphasized.

As in the W12-16 program, two ways of looking at graphs are important: in a detailed way and in a global way. When constructing graphs in a detailed way, the figures on the axes and the scale are important. Constructing global graphs mainly focuses on the shape of the graph. Constructing graphs globally involves sketching the shape based on information presented by the context.

In grade 7, both ways of dealing with graphs are used again. The problem situations involve variables which increase or decrease over time. Various kinds of growth are connected with specific characteristics of the shape of the graph. Simple (recursive) formulas are also used to describe growth. Students get more detailed knowledge about the difference between linear and non-linear growth among which are doubling and halving (exponential growth) and periodic growth. Students recognize linear growth in tables, graphs and formulas.

In grade 8, exponential growth is explored in more depth. Attention is paid to different growth factors and to various kinds of growth. Graphs, tables, and formulas are used to distinguish between linear, quadratic, cubic, and exponential growth.

Student materials and student work: some selected problems

In this paragraph, three aspects of the early algebra of MiC will be illustrated through student material and student work:

- Common sense reasoning in real-life contexts (*Comparing Quantities*, grade 6)

- Constructing Algebraic formulas related to real-life contexts (*Building Formulas*, grade 7)

- Formulas as objects; no connection to real-life contexts (*Patterns and Figures*, grade 8)

Using common-sense strategies

The unit Comparing Quantities offers most problems in a pictorial way or as a story.

Students are challenged to solve these problems, using own strategies. The first example is a simple story with the same numbers as the example from the Dutch textbook of figure 3.

This student work shows exactly (although described in a different way because of the way the problem is presented here) the same reasoning as student strategy 3 in the triangle problem. Starting from the middle (fair division) seems to be a natural strategy for students, as it was already for the Babylonians (Van der Waerden, 1963). Problems like 'the sum of two numbers is 18 and the product is 32' were solved by reasoning from the middle.

In our algebraic notation they create the expression $(9 + x)(9 - x) = 32$ and then $81 - x^2 = 32$ results in $x = 7$. So the numbers are 16 and 2.

The other example from Comparing Quantities is like the 'goldfish-turtle' problem of figure 2

In general there are three different strategies addressed in the unit, all based on informal structures. The

first one ('guess and improve') is a typical student strategy. All strategies of the unit can be translated into formal algebra and represent different aspects of it.

The two examples of student work on '*guess and improve*' in figure 6 show that this strategy is much more sophisticated than 'trial and error'. The first one starts with the real-life experience that a t-shirt is more expensive than a soda and improves from there, while the second one ignores the context and starts mathematically with fair division, working in a systematic way from there on.

A second strategy is called the *exchange* strategy.
In a pictorial way (left in figure 7) or spontaneously using letters (right), both students exchange one t-shirt for one soda and arrive at the price of a single soda. From there they find the price of a t-shirt.

The third strategy is presented in the unit as a method and descibes the several combinations of t-shirts and sodas in a chart.
With only two cells filled, all other entries can be found. In fact this is a kind of mapping of a function of two variables that can be used with linear programming problems. After entering the two number 44 and 30, the diagonal can be filled in up to the borders, with only sodas or t-shirts.

The fourth strategy is also introduced in the unit, in the context of a waiter in a restaurant who uses a notebook to write down the orders of different tables. That is why this is called the *notebook* strategy. The strategy looks the same as the second one (exchange), but there are differences. There is more freedom on manipulating the expressions than there is with the exchange strategy, and this notebook way also works with more than two variables.
The first student stays very close to the unit: the first column shows the tables that ordered something and the prices are in dollars. The second student uses the notebook as an abstract tool for number problems.

Constructing formulas in real-life contexts
In the unit *Building Formulas*, real-life and geometrical situations are used to let students construct own formulas. With well-chosen problems, students often arrive on different formulas that all represent the same phenomenon. The example is taken from the unit and the three student suggestions were found by students in an earlier pilot version of the unit.
The three student suggestions can all be found in the picture by looking at different entities: triangles (Angelina), lengths (David) triangles plus one rod (Maria). Of course, these different representations are a good starting point for discussing equivalent expressions. They all can be transformed into $R = 4L - 1$.
This last formula can also be interpreted from the picture: use as an entity a triangle with the rod on the right-top side of it. Ther are as many entities as the number of rods on the bottom (L) and only the last one

misses the rod on top, so the formula has to be $R = 4L - 1$.

Formulas as objects

In grade 8 formulas are investigated on their own, without any real-life context at all.

Patterns in tables are to be recognized and translated into algebra language. There are also many problems that focus on equivalent formulas. One example from the unit *Patterns and Figures* (grade 8) is shown here to illustrate this kind of activities. The concept of variable is introduced by number strips on which natural numbers represent the variable values of a quantity. So all number strips start with 0.

In this problem, two labeled strips are given: the one of n^2 and the one representing $(n + 2)^2$. After filling in the third one, students have to find the label for this strip. The area diagram of question **d** is used before as a tool that is appropriate to investigate equivalent formulas.

Recently, Java applets are being developed to support the investigations of students on number strips and area diagrams in a dynamic way. These applets can be found on the internet site of the Freudenthal Institute: http://www.fi.uu.nl/wisweb/en/

The applet *2D Geometric Algebra* can be used to investigate equivalent algebraic expressions in which products appear, like $(n + 2)^2$ and $2n$. The full expression can be built up and afterwards split in atomic pieces, that can be rearranged and reconnected. The following four screens show some of the possibilities with the expression $(x + 2)^2$. The text on top of each screen shows the algebraic expression that fits the geometrical shapes in the screen.

In this way, an extra visual and dynamic tool is added for students to investigate algebraic expressions. Students can use it in a way that is much more flexible than the tools that are described in the book. They are free to manipulate expressions in any way they like and can make all the mistakes that will help to improve their insights in algebraic language.

Algebra: contextual or pure?

The teaching and learning of algebra has changed through the projects. Problem situations in real-life settings as well as mathematical contexts are used to make students aware of the usefulness of algebra for solving problems. This mixture of real-life and mathematical contexts made also clear that we have to be careful. In another Dutch curriculum project for vocational education (engineering) it was found that mathematics not always works out in the way that mathematicians think. Two examples will illustrate this. The first example is about inverse functions. In many school books, the inverse function (if it exists) is found by exchanging x and y in the functional equation and then rewrite the equation in the format $y = ...$

So, the straight line through (0,32) and (20,68) has equation: $y = \frac{9}{5}x + 32$ and the inverse function can be

written as $y = \frac{5}{9}(x - 32)$

The two graphs are plotted in the left part of figure 10.

The two pairs of numbers were chosen on purpose: they define the conversion from degrees Celsius to degrees Fahrenheit: 0 °C corresponds to 32 °F and 20 °C is the same as 68 °F. When the variables x and y are replaced by C and F we should get the conversion formulas for Celsius into Fahrenheit. The result is shown in the right side of figure 10. The amazing thing is that there are two different formulas for the conversion of Celsius into Fahrenheit and that is of course not very realistic! What happened is that something was done that is allowed in the world of formal mathematics, but absolutely not allowed in the context of real-life. In the formal system of mathematics, variables have no meaning. In this world the exchange of the roles of x and y is acceptable because they are just placeholders for numbers. In the real world, C and F have a meaning and they cannot be exchanged just like that. What is true in mathematics sometimes is not true in real-life contexts. Thus, the correct second conversion formula should be $C = \frac{5}{9}(F - 32)$. It means that the letters C and F cannot be exchanged. The inverse function cannot be plotted in the same picture, because the names along the axes should then be different for the two graphs. The only way to use the inverse relationship in the graph is to read the information the other way around: from vertical axis to horizontal axis.
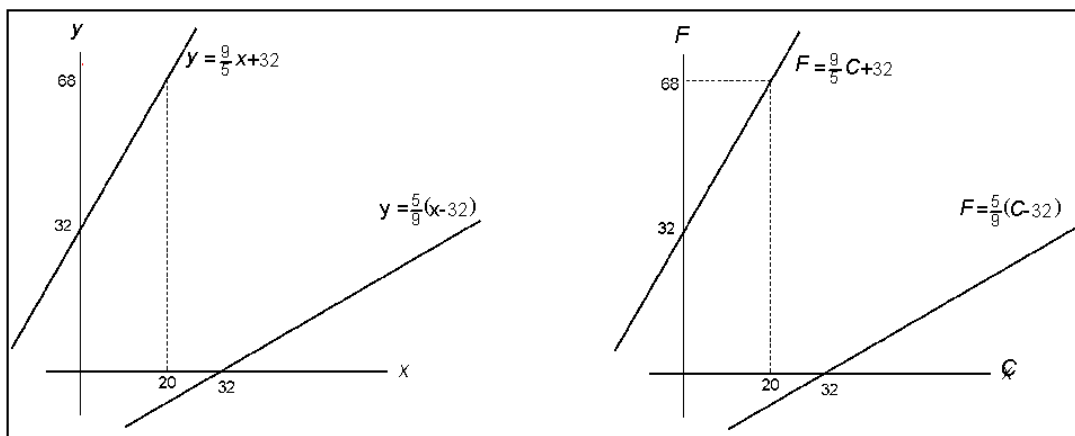

figure 10: inverse functions; true in mathematics (left) and wrong in real-life (right)

The second example is about equivalent expressions. In the world of pure mathematics $2(x-3)$ is equivalent to $2x-6$, but in some applications they can have different meanings.

Two persons A and B walk the same route at the same speed of 2 m/s. B starts 3 seconds after A. The distance walked by them can be described with $D = 2t$ for A and $D = 2(t - 3)$ for B. Recognizable are speed (2) and seconds after (3) for B.

Changing the equation of B into $D = 2t - 6$ creates a new number (6) that is not directly recognizable. The number 6 can be interpreted as the distance that B is behind A when he starts walking. B starts at the same

moment as A (at $t = 0$) and is always 6 m behind A. In the context of the walks, there are two different equations:

$D = 2(t-3)$ has the structure *distance = speed × time*

$D = 2t-6$ has the structure *distance = distance − distance*

Two different graphs can be plotted to support the two different stories:

figure 11: Algebraically the same, but in the context different

Both examples are about the meaning of numbers. In the world of pure mathematics they are just meaningless. But in the world of applications, most of the time numbers are values of quantities and they are connected to units of measurement. We should be aware of the conflicts between the two worlds in order to avoid misconceptions.

Conclusions

Through several curriculum projects, the teaching and learning of (early) algebra has been changed into a more sense-making way, by relating real-life situations and common sense reasoning with the (formal) language of algebra. The MiC project in particular succeeded in designing a program in which the conceptual learning of formal algebra is stretched out over a period of four years, which gives students enough room to get used to both the language and the usefulness of the tool for solving problems. At the same time, we have to be aware of the fact that the mastering of algebraic standard routines in the formal world should never replace common sense strategies. A good balance between formal and informal work is possible. We have to be very clear to students about two different worlds in which algebra is used:

• the world of real-life applications,

where strategies are mostly context-bound and often a mix of context-driven reasoning or calculating and some (basic) algebra,

where the subjects are real entities that make sense and are imaginable,

where a certain freedom is allowed in using own common sense strategies

• the world of formal mathematics,

where standard routines have to be applied,

where the subjects are abstract, meaningless numbers without any value,

where strict rules have to be obeyed.

An algebra program that is built on these two worlds and that makes students able to switch between the worlds when working on problems, can really make 'algebra a tool for solving problems'.

References

Bednarz,N.&C. Kieran (1996) *Approaches to algebra*, Kluwer Dordrecht, The Netherlands, 1996

Gravemeijer, K.P.E. (1990). Globaal Kijken, een kenmerk van algebraïsche deskundigheid.
In:*Nieuwe Wiskrant, tijdschrift voor nederlands wiskundeonderwijs, volume 10, number 2.* Utrecht, The Netherlands: OW&OC. pp. 29-33.

Hoyles, C.& R. Noss. (1998) *Anchoring Mathematical Meanings in Practice.* Presentation at: International Conference on Symbolizing and Modeling in Mathematics Education; Utrecht, The Netherlands, June 1998

Van der Waerden, B.L. (1963) *Science Awakening*, John Wiley&Sons, New York, USA

Van der Kooij, H. (1992) *Matemátika Realista na Holanda.* Educação e Matemátika 23(3), 38-44. Lisboa, Portugal, 1992

Van der Kooij, H. (1999) Reform in Mathematics education in The Netherlands: co-operation of Research and Practice. In: *Relationships between Classroom Practice and Research- CIE- AEM 50.* Salvioni Art Grafice, Bellinzona, Switzerland.

Van der Kooij, H. (2001) Modelling and algebra: How 'pure' shall we be?
In: *Modelling and Mathematics Education. ICTMA9: Applications in Science and Technology*; J.F. Matos, W. Blum (eds), Horwood Publishing, Chichester, England. pp 171-184

Van Reeuwijk, M. (1995) *Students' Knowledge of Algebra*, In: Proceedings of PME 19, Vol 1, Recife Brazil, July 1995, pp 135-150.

Van Reeuwijk, M. (1996) Early School Algebra: a Dutch perspective. Chapter for book on Early Algebra, Kaput (ed) (to be published)

Wenger, R. (1987) Cognitive Science and algebra learning. In A. Schoenfeld (Ed.), *Cognitive science and mathematics education* (pp. 217–251). Hillsdale, NJ: Lawrence Erlbaum.

The Mathematics in Context materials are published by EBEC, Chicago, USA,

website http://www.ebmic.com/mic/common/home.asp