

Productive Struggle – Mining the Power that Lurks within “I Don’t Know”

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Abstract

We can't make our students into seekers if we aren't seekers ourselves. This research-based, practice-oriented paper explores the nature of “desirable difficulties” and the benefits of creating such desirable difficulties to help students shake naïve or loose thinking and to construct “new” knowledge by encouraging transfer of related prior knowledge to new situations. The paper also discusses certain tasks designed to promote the interconnectedness of mathematical knowledge with respect to mathematical concepts from different branches of mathematics and various representations of mathematical concepts.

The intended purpose of this paper is: (1) to help develop instructional strategies that enable all students to engage in reasoning and mathematical discourse about mathematical ideas; (2) to help teachers understand how mathematical ideas interconnect and build on one another to produce a coherent knowledge base of number, geometry, and data analysis, and (3) to recognize and apply mathematics to solve problems, including those in contexts outside of the "apparent" mathematics currently under study.

Introduction

Not everyone can be a mathematician, but everyone can want to be a mathematician. It is important to understand that being good at mathematics is not evidenced by how many answers you know. Instead, being good at mathematics may be best evidenced by what you do when you don't know the answers. We must help students construct and own “new” knowledge in such a manner that they are then able to apply new knowledge in ways that are different from the situation in which it was originally learned.

Albert Einstein may have captured it best in stating that, “Students that can't explain ‘it’ simply, don't understand ‘it’ well enough.”

Why do some students successfully learn mathematics and others do not? Successful students internalize processes, connect concepts and ideas, generalize across domains, and develop personal understanding. According to the Piagetian concept of reflective abstraction, students engage in interiorization, coordination, encapsulation, generalization, and reversal. A curriculum that emphasizes these actions will lead to increased likelihood of connections and transfer.

Most agree that teaching is a complex practice and hence not reducible to recipes or prescriptions. If we can also agree that true learning is reflected by the ability to readily transfer knowledge, skills, attitudes and values from one situation to another, then certain assumptions about mathematics teaching automatically deserve attention.

These assumptions include:

Mathematics is more than a collection of concepts and skills.

A goal of teaching mathematics is to help students develop mathematical power.

All students can learn to think mathematically.

WHAT content is learned is fundamentally connected with HOW it is learned.

Content should be explored at different levels of abstraction.

There are no simple means to assure that these assumptions are adequately addressed in all situations. However, the fundamental task of a mathematician is to convey to students not only what mathematicians know, but also what they do, and how and why they do it.

Body

Sometimes learners express a reluctance to look at mathematics in an alternative way to their initial exposure to the topic. Pleas of "You're going to confuse me!" may actually signal an unrecognized, and certainly unacknowledged level of confusion that is ALREADY present – not potential confusion on the horizon.

There are many benefits to be gained by actually creating something that we'll refer to as DESIRABLE DIFFICULTIES designed to encourage thinking about mathematics as well as enhancing both long-term retention and transfer. Out of apparent chaos and confusion emerges a deeper understanding and appreciation. In fact, students who are successful in making sense of mathematics are those that believe that mathematics makes sense.

Using the sage principle that "No matter what IT is, the chances of finding IT are dramatically increased if you're looking for IT" we must explore techniques to encourage and reinforce mathematics as a way of thinking. The ideas we gather are like so many pieces of colored glass at the end of a kaleidoscope. They may form a pattern, but if you want something new, different, and beautiful, you'll have to give them a twist or two. You experiment with a variety of approaches. You follow your intuition. You rearrange things, look at them backwards, and turn them upside down. You ask "what if" questions and look for hidden analogies. You may even break the rules or create new ones.

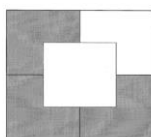
Consider the following incomplete list of students in a class roster:

ANN, Brad, CAROL, Dennis, ???

What name would you propose as the next on this list? Seldom does the question cause "random" answers of "Joe" or Mary" since everyone begins to search for some pattern to apply. Often, the response is along the lines of "Edward" or, perhaps "Edith" – especially those that have self-imposed a rule of alphabetical order. Those suggesting "Edith" may also wish to insist that it should be female to preserve alternating sexes. Then someone chimes in with "Emillie" – with two l's – since the name, whatever it is, should contain seven letters. If the names were also printed in colors, say, black, red, green and blue, then someone else might suggest that the next name should be printed in black to maintain a perceived rotating pattern. Still another would suggest that it should be printed in capital letters. Are we done? How do we know? Have we satisfied EVERY rule? What about the "rule" that calls for three

vowels – did you see that? Should we propose “EVELYNN” instead? Does that violate a “rule” that double n’s should appear in every fourth entry? And so on and so on... How does one ever know if their rule (or combination of rules) is the right rule (or combination)? What makes an answer correct? Is there more than one? Are there infinitely many? You’ll know that you failed in this exemplar if, when finished with the discussion, someone still asks, “But what is the CORRECT answer?” The point here is that it isn’t a mistake to have strong views; it is a mistake to have nothing else.

Some “desirable difficulties” challenge us to consider something from a different perspective than initially considered. Such activities shake the foundations of students that believe conclusions “can’t be altered.” Consider the following graphic as a representation of a fraction. What fraction do you see pictured?



If each student is asked to individually identify a fraction based on this picture you’ll find that many different fractions are offered up for consideration. As a challenge for open-mindedness, if student A “sees” $\frac{3}{4}$ and student B “sees” $\frac{3}{5}$ then challenge each to find the other’s fraction. Many possible fraction representations can be found in this image. Challenge the students to let go of their first impressions and search for something that they didn’t see at first glance. For example, how can this same image also be perceived as $2\frac{1}{4}$?

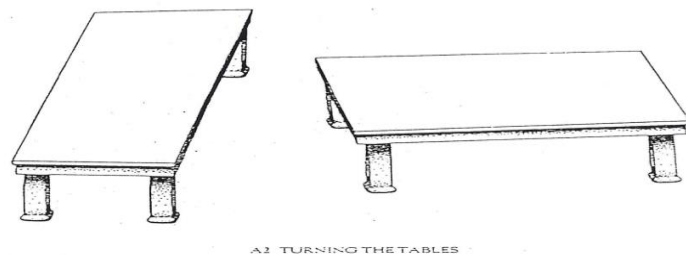
Another example of a “desirable difficulty” is one that involves a situation in which we know WHAT the answer is, we just don’t know WHY that is the answer. For instance, suppose a standard deck of cards is arranged into alternating “red-black” order, and a simple cut of the deck is performed once so that the two “half decks” show alternate colors on the bottom card. Follow this by a simple single shuffle, and ask WHY it is that when the cards are drawn off the top in pairs, there will always be one red card and one black card in each pair? If this is NOT the case, then start over – you’ve done something wrong in the above process. The question is, how can I be so sure? WHY must this always be the case? Why aren’t there any pairs of red or pairs of black? Isn’t a shuffle SUPPOSED to mix the cards?

A common category of “desirable difficulties” is represented by paradoxes. Unlike most DD’s, the main effect of a paradox is that it leaves you scratching your head looking for an explanation. Still, these illustrate the dilemma of unexpected results. A more rewarding cousin of the paradox is the “desirable difficulty” that has a twist or surprise ending.

Consider the number of segments of length “1” that can be found on a standard 5x5 peg geoboard. A moment’s reflection yields the answer of 40 (20 horizontally and 20 vertically). How many segments of length 2? 30 (15 horizontally and 15 vertically). How many of length 3? 20 (10 horizontally and 10 vertically). How many of length

4? 10 (5 horizontally and 5 vertically) So, how many of length 5 on a standard 5x5 geoboard? The pattern shouts for 0, but, in reality, the answer is 8.

One of the most common exemplars of a “desirable difficulty” may be found in an optical illusion. While such illusions neither hold the attention nor have the rich extensions that subsequent exemplars might have, they, nevertheless, provide the reader with a quick emotion of incredulous surprise. For example, consider the following illustration.



Despite the appearance, the two tabletops are congruent! I'll leave it to the reader to cut a shape to match the table on the left and then rotate it to the table on the right.

The feeling of “impossible” or “that can't be” stays with the reader even after having “proven” the congruence for him/herself. What's missing, though, is the “so what?” of a richer learning experience.

Concepts, terminology, and symbols are the foundations on which mathematics learning and linkages are built. We learn new concepts, terminology, and symbols in many ways by exploring information that is presented to us in many forms. The notion of "having learned something" implies, among other things, that the learner can readily demonstrate the ability to identify, label, use, and transfer the knowledge, skills, and processes learned. While linking mathematics to “real-world” experiences is an effective method of introducing new concepts and emphasizes the "use and transfer" components of learning, student verbalization of connections through desirable difficulties is another significant means of assisting with the "identification and label" components of transfer.

Student exploration emphasizes opportunities for internalizing and anchoring information by having students verbalize – not blindly, but meaningfully - important

facts, identifications, definitions, and procedures. Learners internalize information through continued exposure and concerted effort. Students do not learn by doing – they learn by thinking about what they are doing. Mathematics, therefore, is best characterized as a series of action verbs rather than as the rules that those action verbs produce. Students must engage in Modeling, Analyzing, Thinking, Hypothesizing, Experimenting, Musing, Applying, Transferring, Investigating, Communicating and Solving. These practices anchor information and help students absorb and retain the information upon which critical thought in MATHEMATICS is based.

A healthy combination of student action and linking new material to previously learned mathematics concepts, procedures, and practical experiences will set the stage to help students feel more comfortable in their knowledge and understanding of new concepts or procedures. Understanding gained through concept development and linkages, in combination with meaningful memorization of important terms, concepts, and algorithms, gives students power over mathematics. This power leads to confidence and an increased comfort level in their ability to function and reason mathematically.

Linking new material to previously learned mathematics concepts, procedures, and practical experiences sets the stage to help students feel more comfortable in their knowledge and understanding of new concepts or procedures. Mathematics teachers are cognizant that the concepts and skills they teach today are often used later as building blocks for more abstract ideas. It is just as important to be aware of the benefits of using these links as a means of building a spirit of partnership in learning.

It is as important for us to acknowledge as it is for students to realize that they can do the mathematics that we are about to teach -- before we teach it. Students that believe they possess prerequisite knowledge that leads to a "new" concept play a more active role in learning than those who feel the teacher is the sole provider of the knowledge needed to learn a new concept.

Introducing concepts through linkages enables students to relate new ideas to a context of past learning. Students are then more likely to understand and, therefore, absorb new material. For example, students that are being taught to multiply polynomials should be led to the connection between this algorithm and the standard algorithm for multiplying whole numbers taught in the third or fourth grade. Similarly, the student who recognizes "lining up decimal points when you add decimals" as just an extension of "adding like place values from whole numbers" is far ahead of the student who sees this generalization as a NEW rule to learn, unrelated to prior knowledge.

Connecting mathematics to real-world experiences is another effective method of introducing new concepts through linkages. While students too infrequently link their transactions at the store to mathematics class, they often quickly understand that if one candy bar costs fifty cents, then two will cost a dollar. Thus, buying candy at a store can be linked to such mathematics concepts as ratios, proportions, ordered pairs, linear graphs, patterns and functions. But such "simple" links must also be followed by less direct paths to help students find their way to the richness of connections. For example, how might a telephone book help develop an approximation for pi?

Consider this chain of related concepts:

- Subtraction as a difference $[x_1 - x_2]$
- Distance between two points, x_1 and x_2 , on a number line
- Distance between two ordered pairs on a grid
square root of $[(x_1 - x_2)^2 + (y_1 - y_2)^2]$
- Pythagorean theorem $[a^2 + b^2 = c^2]$
- Equation of a circle $[x^2 + y^2 = r^2]$
- Area of a circle $[A = \pi r^2]$
- Monte Carlo probability model $[P(e) = P(A_c)/P(A_s)]$

$$\text{Since } \frac{P(\text{inside circle})}{P(\text{inside square})} = \frac{\pi r^2/4}{r^2} = \frac{\pi}{4}$$

We can then use the last four digits of 200 random numbers in the telephone book. Split each of these into two two-digit numbers. Treat the first pair as “x” and the second pair as “y” and then, Determine how many phone numbers out of the 200 selected yield $x^2+y^2 \leq 10000$ and, using the probability formula, we have an approximation for pi from a telephone book!

Conclusion

Understanding gained through concept development and linkages, in combination with memorization of basic facts and algorithms, gives students power over mathematics. This power leads to confidence and an increased comfort level in their ability to function and reason mathematically. Students exposed to mathematics in this manner are more likely to have the ability to set up problems, not just respond to those that have been previously identified for them. They are more likely to value a variety of approaches and techniques; to have an understanding of the underlying mathematics in a problem; to have the inclination to work with others to solve problems; to recognize how mathematics applies to both common and complex problems; to be prepared for open problem situations; and to believe in the value and utility of mathematics.

Ideally, spending valuable instructional time on desirable difficulties will allow us to better address the expectations of a school mathematics program. As teachers, it is important to recognize that our beliefs about the nature of mathematics influence what students learn and what they perceive mathematics to be. Providing students with a rich array of opportunities to construct meaning from experiences that challenge the ways in which student think and act when confronted with unfamiliar mathematics enhances the ability to make the conceptual leap from concrete to abstract reasoning.

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