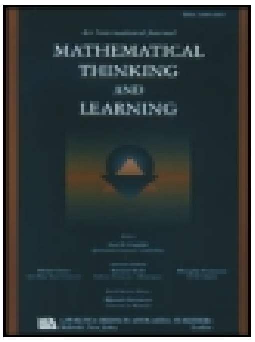

Mathematical Thinking and Learning

ISSN: 1098-6065 (Print) 1532-7833 (Online) Journal homepage: http://www.tandfonline.com/loi/hmtl20

Mathe-Didactical Reflections on Young Children's Understanding and Application of Subtraction-Related Principles

Marja van den Heuvel-Panhuizen & Adri Treffers

Mathematical Thinking and Learning, 11: 102–112, 2009

ISSN: 1098-6065 print / 1532-7833 online
DOI: 10.1080/10986060802584046

Mathe-Didactical Reflections on Young Children's Understanding and Application of Subtraction-Related Principles

Marja van den Heuvel-Panhuizen
Fisme, Utrecht University, the Netherlands; IQB, Humboldt University Berlin, Germany

Adri Treffers
Fisme, Utrecht University, The Netherlands

In this article we react to the studies in this special issue of *Mathematical Thinking and Learning* on young children's understanding and application of subtraction-related principles. We discuss the results of these studies and the problems presented to the children from a mathe-didactical point of view; including both the perspective of the mathematical content and of its learning and teaching. The importance of this special issue is that it brings together psychologically and didactically oriented research.

INTRODUCTION

This special issue of *Mathematical Thinking and Learning* touches on the core of young children's learning to operate with numbers. The studies reported here delve into the foundation that supports operating with numbers and make it clear that learning to calculate is not just a matter of learning a particular calculation procedure, but that it requires an understanding of number relationships and properties of operations. When using this understanding, calculating is not just a case of knowing the counting sequence and having a good memory but also one of thinking. This means that *mathematics* enters into *arithmetic*. This mathematical deepening of calculation can manifest itself in choosing an effective strategy. Consider two examples in which the thoughtful choice of a strategy can significantly reduce computational effort. Determining the outcome of 61 + 24 – 23 can be simplified by recognizing that 24 can be decomposed into 1 + 23, which converts the problem into 61 + 1 + 23 – 23. As the addition and subtraction of 23 cancel each other, a student is left with the relatively simple sum of 61 + 1. The subtraction problem 782 – 179 can be simplified by the adding-on process: adding 3 to the subtrahend 179 to get to 182 and then adding 600 to this partial sum get to 782. The principles that lie behind these shortcuts have been investigated in the studies included in this special issue.

Correspondence should be sent to Marja van den Heuvel-Panhuizen, Fisme, PO Box 9432, 3506 GK Utrecht, The Netherlands. E-mail: m.vandenheuvel@fi.uu.nl

In our contribution we react to the studies included in this special issue from the perspective of didactics as a scientific discipline (Biehler, Scholz, Strässer, & Winkelmann, 1994; Freudenthal, 1978; Niss, 1999; Treffers, 1987; Wittmann, 1984). We base our reaction on a *mathe-didactical* analysis of the problems provided to the children. The term mathe-didactical analysis as it is coined by Van den Heuvel-Panhuizen and Teppo (2007) includes the perspectives of the subject matter (“mathe”) and that of the learning situations and classroom teaching of the subject matter (“didactical”). This means that we examine the mathematical meaning of the investigated problems and the underlying principles and ask ourselves why it is important that children learn them and how that learning can be stimulated.

STUDIES ADDRESSING THE INVERSE PRINCIPLE

The studies of Nunes, Bryant, Bell, Evans, and Hallett (this issue); Bisanz, Watchorn, Piatt, and Sherman (this issue); Gilmore and Papadatou-Pastou (this issue); and Baroody, Lai, Li, and Baroody (this issue) all address whether children understand the inverse principle—can immediately recognize that adding a number b to a number a can be undone by subtracting b ($a + b – b = a$) and vice versa ($a – b + b = a$). A shortcut task was used in the first three studies. This entailed comparing children’s scores on inverse problems (e.g., $a + b – b = ?$) with those on control problems (e.g., $a + b – c = ?$).[1] Theoretically, the former do not require calculation (if one understands the inverse principle) whereas the latter do. Whereas the starting amount and the transformations are quantifiable with the shortcut task, Baroody et al. (this issue) used algebraic-reasoning tasks in which the starting number is unknown (e.g., $x + 2 – 2$ and $x + 3 – 2$) and children are asked if the starting number has changed and, if so, how.

The main contribution of these four studies to the body of knowledge about young children’s learning of mathematics is that they have explored in a systematic and fine-grained way the cognitive roots of children’s understanding of the inverse principle by addressing several important pending questions, such as (a) whether understanding of this principle has a qualitative or a quantitative basis, (b) whether dealing with approximate quantities plays a role in developing an understanding of quantitative inversion or that exact quantification methods are necessary, and (c) whether calculation procedures or concepts develop first. Hereafter we will discuss their answers to these three questions. But before doing so, we use some general concerns from a mathe-didactical perspective to raise some issues that need to be taken into account in future theoretical reflection and empirical research on the inverse principle.

Issues that Raise Mathe-Didactical Concerns

First, inverse items such as 8 + 7 – 7 are not necessarily a special case or in a category different from items such as 8 + 6 – 5 (control problems). As with inverse items, mathematical principles can be applied to control items. That is, conceptually based shortcuts can be applied to control items and inverse items. In 8 + 6 – 5, for example, decomposing the 6 can convert the problem to

[1]In the study by Nunes et al. (this issue) the control problems are not of the type $a + b – c$, but of the type $a + a – b$. Moreover, their inverse problems also include problems of the type $a + b – (b +/– 1)$.

8 + 1 + 5 – 5. Similarly, Bisanz et al. (this issue) observed that Bryant, Christie, and Rendu's (1999) item 24 + 10 – 9 could be solved by converting it into 24 + 10 – 10 and then adding 1 to 24. Although such principled reasoning on control items can reduce the reaction time and accuracy difference between such items and inverse items (and thus perhaps add to measure error), it probably does not entirely eliminate these differences for two reasons. One is that the application of the inverse principle in the inverse problems requires one step whereas this requires two or more steps in the vast majority of control items. Another reason is that the inverse principle is relatively salient compared with other principles that might be applied to control items. Clearly, investigators need to consider carefully their choice of control items and what other principled knowledge their participants might possess and apply.

A second point of concern relates to the distinction between inverse problems ($a + b - b = ?$) and complement problems ($a + b = c; c - b = ?$). The authors of the introduction to this special issue (Baroody, Torbeyns, & Verschaffel, this issue) have suggested this terminology for the sake of consistency and clarity, and it has been accepted by all contributors. We agree with this pragmatic solution, but we doubt whether it is warranted to speak of two types of problems. From a mathe-didactical perspective, the distinction may be more of a notational nature rather than a difference in underlying mathematical structure. As Figure 1 illustrates, the three notations all entail first adding *b* to a number and than subtracting *b*.

The question is whether problems that have the same mathematical structure can serve as an operationalization of two different mathematical principles. The results found by Nunes et al. (this issue) are interesting in this respect. Their prediction that complement problems—the authors call these problems "transfer complement problems"—are more difficult than inverse problems and that an intervention that improves children's understanding of inversion should also increase their success with complement problems, was not confirmed. To the researchers' surprise the complement problems (which consisted of two problems, each including one operation) were actually easier than the inverse problems (which have two operations in one problem). Although this result does not answer the above question, it gives an inkling that the problems that were used to measure the inverse principle could be improved on the point of how the problems are notated. Nunes et al. (this issue) also acknowledge that one possible explanation for the complement problems being the easier of the two might lie in the format of the problems: "Young children are more used to questions with one operation; two operations in sequence, in a single item, are quite unusual, at any rate in the classroom." We therefore suggest in order to assess children's understanding of

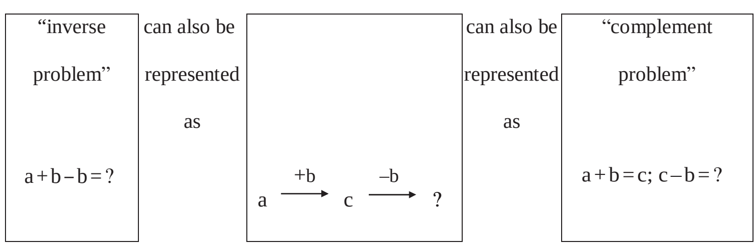

FIGURE 1 Three different notations.

the inverse relationship between addition and subtraction by presenting them inverse problems that each consist of two number sentences each containing one operation. This solution would also solve another problem we have with inverse problems in which the + *b* and the – *b* are in one number sentence.

Third, in number sentences consisting of several operations, the computational priority rules might cause confusion. The best way to avoid uncertainty for the children about the order in which the operations should be performed in such number sentences is by using parentheses.[2] However such a notation is very unusual for primary school children.

A fourth point of concern is the trouble children might have with problems having a subtraction at the start. If the first number is smaller than the second, e.g., 5 – 7 + 7, children are confronted with an "unsolvable" problem because 5 – 7 does not have a solution for them.

Qualitative or Quantitative Basis?

As explained by Bisanz et al. (this issue)—while referring to Sherman and Bisanz (2007)—two main hypotheses exist in connection to this question. The first hypothesis assumes generalization from qualitative thinking. This means that qualitative inversion in which a previous transformation is cancelled out and that is grounded in everyday, non-quantitative experiences is seen as the basis of quantitative inversion. To clarify what this qualitative inversion includes, Bisanz et al. (this issue) and Nunes et al. (this issue) mention the example of a shirt that is first dirtied with mud and then cleaned. Referring to Klein and Bisanz (2000), Gilmore and Papadatou-Pastou (this issue) see give-and-take games as a context for developing qualitative inversion. If this first hypothesis is correct, then children would show qualitative inversion prior to quantitative inversion.

The second hypothesis includes induction from quantitative experience. This hypothesis means that quantitative inversion is constructed on the basis of children's counting and calculation experiences (when the same quantity is added to and then subtracted from a set, the initial and final set sizes are the same). This second hypothesis implies a link between success on inverse problems and counting and calculation skills. With the exception of one study, the meta-analysis of Gilmore and Papadatou-Pastou (this issue) did not find evidence for a reliable group of children with good calculation skills but poor understanding of inversion. However, Baroody et al. (this issue) found a connection between success on inverse problems and the ability to quantify accurately and reliably small collections. They even think that qualitative experience can provide contradictory or confusing precedents for operations on numbers.

The conclusion drawn by Bisanz et al. (this issue) is that both hypotheses cannot be supported convincingly by empirical data. On the one hand, they refer to studies that showed that toddlers did not perform better on problems affording qualitative solutions than on those affording solely quantitative solutions. On the other hand, they conclude that, in general, students' performance on inverse problems and their competence in counting or calculation

[2]In the case of 15 – 7 + 7, using parentheses, (15 – 7) + 7, is the only way to avoid misunderstanding if one keeps one number sentence. Another option is to use an arrow notation:

$$15 \xrightarrow{-7} \ldots \xrightarrow{+7} ?$$

are not well correlated. These contradictory research results indicate that further research is needed to sort things out between these two hypotheses. Maybe both hypotheses are wrong and we have to look for a more adequate explanation, but another possibility is that one of the hypotheses is, in fact, correct and the problem lies in the studies carried out to test these hypotheses.

Approximate or Exact Quantification Methods as a Basis?

A question that follows from the quantitative-basis hypothesis is whether for a quantitative inversion, precise quantities must be considered. Here too, the authors of the four articles reach different conclusions. Gilmore and Papadatou-Pastou (this issue) clearly take the view that children can recognize inverse relationships with approximate representations of quantities before they can do so for exact symbolic quantities. This means that they may develop understanding of quantitative inversion relating to exact numbers out of a pre-existing ability to recognize this relationship in terms of approximate numerosities. However, Baroody et al. (this issue) go against this conclusion. Although they do admit that inexact representations and processes might help pave the way for subtraction concepts, they assume that exact quantification methods are needed to induce a reliable and general understanding of the negation ($n - n = 0$), identity ($n - 0 = n$), and inversion principles. In this, they see the negation and the identity principles as a basis for understanding inversion. Bisanz et al. (this issue) also think that for a quantitative inversion precise quantities must be considered.

For the inverse principle, it seems that Baroody et al. (this issue) and Bisanz et al. (this issue) have a good case. In fact, cancelling out only works when the numbers are exactly the same. However, for the complement principle (which underlies the solution of problems like $a - b = ?$ problems by indirect addition) this may be different. An indication for this may be the powerfulness of the idea of the “nearly disappearing problems” (Menne, 2001): Asking children for an estimate of the result of a problem like 61 – 58 = ? prompted them to add on from *b* until *a* was reached.

Concepts or Procedures First?

In addition, when asking about the relation between understanding the concept of inversion and being able to perform calculation procedures, there is a connection to the question of whether there is a qualitative or quantitative basis for understanding inversion. In particular, Gilmore and Papadatou-Pastou (this issue) explicitly examine this concept-procedure question. They refer to Gilmore and Bryant (2006), who found that all situations are possible, including good understanding of inversion and good calculation skills, poor understanding of inversion and poor calculation skills, and good understanding of inversion despite poor calculation skills. Bisanz et al. (this issue) mention Jack, who has proficiency at calculation but not in solving inverse problems. The meta-analysis carried out by Gilmore and Papadatou-Pastou (this issue) shows that in a vast majority of studies there were children with good understanding of inversion despite poor calculation skills. Although this finding might indicate that children can develop conceptual understanding in the absence of proficient procedural skills, this conclusion should be treated with caution because in 80% of the studies included in the meta-analysis of Gilmore and Papadatou-Pastou (this issue) the investigated children are close to the age of 7 years or older, which is rather old to answer the question whether concepts or procedures develop first.

Contribution to Mathematics Education

Although the four articles contribute to our knowledge of the cognitive roots of children's understanding of the inverse principle as defined and operationalized in the present special issue, we doubt the usefulness of this rather strict interpretation of inversion for mathematics education. This uncertainty can, for instance, be explained by means of an example given by Nunes et al. (this issue). According to these authors, knowing that 9 + 1 – 1 = 9 might help children to understand why adding nine to a number gives the same result as adding ten and then taking one away. In other words, when a child has to calculate 37 + 9, he or she might think of 37 + 9 + 1 – 1, then take 9 + 1 together and come to 37 + (9 + 1) – 1 or 37 + 10 –1. We are not convinced that this is the trajectory in which children actually develop the ability to make such calculation shortcuts. Our experience as researchers of learning and teaching mathematics is that children arrive at this smart strategy in a more direct way, namely by realizing that "+ 9" is "almost + 10". When they are working on an empty number line, for instance, they can think of first making a jump of ten and then correcting that action by jumping one back (see Figure 2).

Another concern comes from the results found by Nunes et al. (this issue) that the children who were taught inverse problems ($a + b – b$) did not improve in the control problems ($a + b – c$) that require calculation. Of course, this is in agreement with how inversion is operationalized in their research and the distinction that is made between problems that do not need calculation and those that do (see the section: "Issues that Raise Mathe-Didactical Concerns"), but this result also makes it clear that understanding inversion strictly as being able to solve an arithmetic problem in which + b is followed by – b in itself contributes little to learning to calculate. Furthermore, if the control problems were not solved better, one could conclude that nothing has come of the calculation shortcuts that understanding of the inverse principle was supposed to support.

STUDIES ADDRESSING THE COMPLEMENT PRINCIPLE

We have left the article by Torbeyns, De Smedt, Stassens, Ghesquière, and Verschaffel (this issue) for last, since their studies target the understanding of the complement principle—in particular the use of indirect addition to solve subtraction problems. Compared with the other four studies, Torbeyns et al.'s (this issue) studies are more didactical in character. They make a direct link to the teaching of mathematics, and as a result their studies are more closely related to our work as mathematics educators. While commenting on their results, we will also bring in some of our own findings.

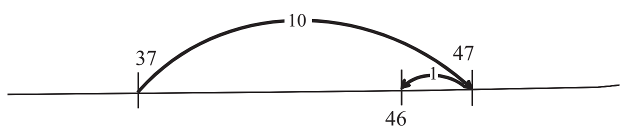

FIGURE 2 Solving 37 + 9 on an empty number line.

One of the core goals of today's mathematics education is that children learn to calculate. In addition to finding the right answers, they should find these answers in an efficient way by choosing appropriate strategies and tools. How a calculation is carried out depends on the one hand on the ability level of a student—which here means the student's knowledge of numbers and the repertory of strategies he or she has available—and on the other hand on the nature of the numbers involved and the context of the problem.

The most important message that has emerged from the studies of Torbeyns et al. (this issue) is that children in primary school seldom make use of the indirect addition strategy for two-digit subtractions with a subtrahend close to the minuend. This is an alarming result that has to be addressed if we want to achieve the above-mentioned goal of being able to calculate efficiently. A first requirement is to know exactly when children do or do not apply indirect addition when solving subtraction problems and what conditions can contribute to strengthening its use. Through a systematic exploration in a cycle of three studies, Torbeyns et al. (this issue) have already mapped much of the use of the indirect addition strategy. At the same time, some points remain that from a mathe-didactical perspective require a deeper understanding, such as (a) the broad constitution of subtraction, (b) the role of the number line, (c) the difference between bare number problems and context problems, and (d) what teaching can contribute to learning to use indirect addition to solve subtraction problems.

Broad Constitution of Subtraction

Solving a subtraction problem by indirect addition means that the problem is solved by adding on from the subtrahend. This is an efficient strategy when the subtrahend is close in size to the minuend. Instead of counting back or subtracting a large number, you can count on or add a small number. However, apart from the nature of the numbers there are more reasons to calculate a subtraction as an addition and so make use of the complement principle. This has to do with the two phenomenological appearances of subtraction: subtraction as *taking away* and as *determining the difference*. This second appearance is based on a comparative action in which the difference can be found by taking away (direct subtraction) and by bridging the distance, for example, from the smaller number to the larger number by adding on (indirect addition).

In the first appearance—that of subtraction as taking away—the only matching action is that of taking away. This interpretation of subtraction is too one-sided. As Freudenthal (1983, p. 107) already emphasized in his didactical phenomenological analysis of subtraction "explicit taken away suffices as little for the mental constitution of subtraction as uniting explicitly given sets suffices for addition". However, the studies by Torbeyns et al. (this issue) do not comprise these two interpretations of subtraction. Because they are based on bare number problems they will likely prompt subtraction as taking away; i.e., following the literal meaning of the minus sign. If Torbeyns et al. (this issue) had also given occasion to consider subtraction as determining a difference, it would have been more likely that indirect addition would have appeared.

The Special Role of the Number Line

The model to support this "two-way traffic" (taking away and adding on) is the number line. Again it was Freudenthal (1983, p. 107) who pleaded for using what he called, the "geometrical concreteness of the number line" in which the two methods connected to the two interpretations of subtraction can be observed, namely "taking away at the start" and "taking away at the end" (see also Menne, 2001; Veltman, 1993; Veltman & Treffers, 1995).[3] The first method reflects the adding on strategy. According to Freudenthal (ibid.), "[c]hildren learn quickly which method is more useful in each particular case: if the subtrahend is smaller than half the minuend it is taken away at the end, otherwise at the start".[4] Later on, Veltman (1993) showed how helpful the number line is for making children aware of the two strategies and for choosing the most efficient strategy for doing subtraction problems.

The Difference Between Bare Number Problems and Context Problems

In addition to what model is used to support the children's calculating, the way subtraction problems are presented is also of eminent importance for the strategy that the child will use. First, one can think of the influence exerted by the action that is described in the context of the problem. Eating candy refers to taking away and finding out how many pages still have to be read refers to adding on.

However, even without this explicit strategy guidance, contexts influence whether indirect addition is used. For instance, Van den Heuvel-Panhuizen (1996) found remarkable performance differences between context problems—in which neither the read-out text nor the pictures referred to "take away" or "subtract"—and the subtraction problems presented as bare number problems. Both types of problems were included in the same tests. The context problems were about comparing the number of beads in two jars (33 – 25), finding the right change (100 – 85), and comparing the heights of two boys (145 – 138). In addition, the tests also included a pair of addition problems. The context of the addition problem (67 + 23) was about finding the total price of a set of jogging pants and jacket. Table 1 shows the results for the pairs of problems.[5]

All subtraction problems presented in context yielded higher scores than the bare number versions of these problems, although the calculations to be carried out were the same. The results found for the addition problem indicate that it was not the motivational effect of the context that caused the difference. The biggest distinction between the addition problem and the subtraction problems is that for the addition problem there is nothing to gain from the perspective of strategy use—the context did not elicit an easier strategy to carry out the addition—while in the subtraction problems the context opened up the indirect addition strategy. In the bare number format this obviously did not happen. The minus sign is probably too strongly connected to "taking away"—which is equally true for minus words (like "lost" or "gave away" or "fewer") in word problems—whereas pictures allow more room for interpretation. Because of this dominant feature of the minus sign it is no wonder that Torbeyns et al. (this

[3]Note that there is a remarkable difference between the *structured* number line that was suggested by Freudenthal and included in the Wiskobas program in the 1970s and the *empty* number line that was suggested by Treffers in the mid-1980s.

[4]Although we do not agree with Freudenthal's criterion of the size of the difference between the minuend and the subtrahend, his definition falsifies Torbeyns et al.'s (this issue) conclusion that didactical publications do not explain what is precisely meant by a small difference.

[5]In the case of the context problem 145 – 138, it was not possible to include a similar bare number problem in the test because at the time of testing the children were not familiar with bare number subtraction problems beyond 100, which need borrowing.

TABLE 1
Percentage Correct for Context and Bare Number Problems
(van den Heuvel-Panhuizen, 1996)

		Percentage Correct	
Problem	*Format*	*April/May Grade 2* *n = 432*	*September Grade 3* *n = 425*
33–25	Context	64%	71%
33–25	Bare number	40%	49%
100–85	Context	60%	70%
100–85	Bare number	55%	54%
145–138	Context	52%	60%
67+23	Context	74%	80%
67+23	Bare number	75%	81%

issue) find less evidence of the use of the indirect addition strategy, even in the study in which the children were explicitly asked to think up an alternative strategy.

What Teaching is Needed?

The importance of the studies by Torbeyns et al. (this issue) is that they demonstrated that all our didactical knowledge about the potential of the complement principle to make subtraction problems easier is not automatically capitalized in classroom practice. It is indeed a rather hard confrontation to have to discover that even the children from the classes that were taught indirect addition in Torbeyns et al.'s (this issue) third study did not exhibit efficient and flexible use of this strategy. The crucial question then is why this is the case. Torbeyns et al. (this issue) have themselves given an answer that we fully support. We too believe that the teaching in this study was not sufficient to bring the students to an adequate use of the strategy. Klein's study (1998) is in our view the best proof that it is possible that children learn this. Moreover, Klein's study showed that the use of this strategy also supports weak students. So, using indirect addition is not restricted to high achievers.

Torbeyns et al. (this issue) indicate what powerful learning environments must look like to develop the adaptive expertise of children so that they use indirect addition in an adequate way to solve subtraction problems. From our own experiences and didactical insights we recommend special attention for:

- widening the interpretation of subtraction (subtraction as taking away and subtraction as determining the difference) as well as a broader view of strategies; not just contrasting the use of indirect addition to the direct subtraction strategy but also involve other strategies that are helpful for subtraction like compensating
- including both bare number problems (even with offering new room for the traditional open number sentences) and context problems; the latter should go beyond the context problems with missing addend structure as proposed by Torbeyns et al. (this issue), and certainly should not be limited to word problems

- not using a set of rules for applying the indirect addition strategy; apart from the fact that this—as Torbeyns et al. (this issue) indicate in their conclusion—does not fit well with fostering flexibility in strategy use, an extra decision algorithm could raise the cognitive load and thus be counterproductive
- use of the empty number line
- developing questions such as “About how much left?” that trigger the use of indirect addition
- discarding the opinion that children get confused by indirect addition
- attention for the didactical contract: Do the children know that they may add on instead of subtract?[6]

Concerning the final point, it is interesting that most university students in the first study of Torbeyns et al. (this issue) did apply the indirect addition strategy. They most likely did not feel as tied to the unwritten rule that says “*plus* means adding and *minus* means subtracting.” Furthermore, attention for the didactical contract also implies being attentive to the possibility that children have strategies of which they are not aware or of which they want to keep secret.

CONCLUDING REMARKS

We are pleased that the editors took the initiative to give space to a didactical perspective on learning subtraction-related principles in a special issue that is dominated by psychologically oriented research. We see broadening the view and including mathe-didactical knowledge as a step forward in the understanding of children’s learning of mathematics. We agree when Gilmore and Papadatou-Pastou (this issue) say: “[W]hen describing how children develop understanding of mathematical concepts, it is important to consider multiple ways in which children may come to this understanding. The challenge for theorists from both cognitive psychology and mathematics education is to account for the variety of ways in which children might discover and integrate conceptual knowledge with their knowledge of mathematical procedures.” In our view, the recommendation might go further than keeping open multiple ways of learning mathematics. In fact, every study into the learning of mathematics should include the basis of a mathe-didactical analysis (Teppo & Van den Heuvel-Panhuizen, 2007). In addition—as this special issue has shown—domain-specific didactics can take advantage of results from psychological studies. It is therefore valuable that both disciplines could meet in this special issue and have a debate on the learning and teaching of young children’s understanding and application of subtraction-related principles. How this research is conceptualized and carried out within these two disciplines would be an interesting topic as well, but we will save that discussion for another opportunity.

REFERENCES

Biehler, R., Scholz, R. W., Strässer, R., & Winkelmann, B. (Eds.). (1994). *Didactics of mathematics as a scientific disciplin*e. Dordrecht: Kluwer Academic Publishers.

[6]Our experiences with strategy differences between girls and boys have taught us that girls often think that the sign determines what operation has to be carried out.

Bryant, P., Christie, C., & Rendu, A. (1999). Children's understanding of the relation between addition and subtraction: Inversion, identity, and decomposition. *Journal of Experimental Child Psychology*, *74*, 194–212.

Freudenthal, H. (1978). *Weeding and sowing. Preface to a science of mathematical education*. Dordrecht: Reidel Publishing Company.

Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures*. Dordrecht/Boston/Lancaster: D. Reidel Publishing Company.

Gilmore, C. K., & Bryant, P. (2006). Individual differences in children's understanding of inversion and arithmetical skill. *British Journal of Educational Psychology*, *76*, 309–331.

Klein, A. S. (1998). *Flexibilization of mental arithmetic strategies on a different knowledge base: The empty number in a realistic versus gradual program design.* Utrecht, the Netherlands: CD-β Press/Freudenthal Institute, Utrecht University.

Klein, J. S., & Bisanz, J. (2000). Preschoolers doing arithmetic: The concepts are willing but the working memory is weak. *Canadian Journal of Experimental Psychology-Revue Canadienne de Psychologie Experimentale*, *54*, 105–116.

Menne, J. J. M. (2001). *Met sprongen vooruit. Een productief oefenprogramma voor zwakke rekenaars in het getallengebied tot 100—een onderwijsexperiment*. [Jumping ahead. A productive program for weak learners in the number domain up to 100. A teaching experiment.] Utrecht, CD-β Press/Freudenthal Institute, Utrecht University.

Niss, M. (1999). Aspects of the nature and state of research in mathematics education. *Educational Studies in Mathematics*, *40*, 1–24.

Sherman, J., & Bisanz, J. (2007). Evidence for use of mathematical inversion by three-year-old children. *Journal of Cognition and Development*, *8*, 333–344.

Teppo, A., & Van den Heuvel-Panhuizen, M. (2007). *Mathe-didactical analysis: A crucial component of task design*. Paper presented at Mathematical Thinking: An Interdisciplinary Workshop University of Nottingham, November 21–22.

Treffers, A. (1987). *Three dimensions. A model of goal and theory description in mathematics instruction—the Wiskobas Project*. Dordrecht: D. Reidel Publishing Company.

Van den Heuvel-Panhuizen, M. (1996). *Assessment and realistic mathematics education*. Utrecht, the Netherlands: CD-β Press/Freudenthal Institute, Utrecht University.

Van den Heuvel-Panhuizen, M., & Teppo, A. (2007). Tasks, teaching sequences, longitudinal trajectories: about micro didactics and macro didactics. In J. H. Woo, H. C. Lew, K. S. Park, & D. Y. Seo (Eds.), *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, p. 293). Seoul: PME.

Veltman, A. (1993). *Van het begin of van het eind.* [From the beginning or from the end.] Master thesis. Utrecht University.

Veltman, A., & Treffers, A. (1995). Midden tussen de getallen. [In between two numbers.] *Willem Bartjens*, *2*, 20–23.

Wittmann, E. (1984). Teaching units as the integrating core of mathematics education. *Educational Studies in Mathematics*, *15*, 25–36.