Functions, Covariation, and the Cartesian Connection: Examining Two Learning Trajectories for Pre-Service Secondary Teachers

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supported by the NSF and the EAF

Professional development for mathematics teachers often combines challenging inquiry with a close examination of student work. But how do these activities relate? Do they support each other? In this talk we examine hypothetical learning trajectories in a course for pre-service secondary teachers where it was believed certain investigations would enhance their abilities to make sense of K-12 students mathematical work on related topics. We will discuss on what worked and what didn't, what modifications in the contexts and problems helped, and what this tells us about preparing undergraduate math majors for secondary teaching.

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I. Overview and Background

Today's talk describes learning trajectories in a course for prospective secondary teachers. It begins by describing what this course is about and then lays out the obstacles students in the course confront the trajectory is supposed to address. Then student work on an assessment at the end of the instructional sequence is examined to illuminate what the instructional unit is trying to accomplish. The talk closes by considering the two hypothetical learning trajectories and was learned implementing them.

- I. Overview and Background
- II. Guiding Assumptions, Features of the Course, and the Research Question
- III. Functions and Covariation, what are we after here?
- IV. Student work on an assessment at the close of the instructional sequence.
- V. Two Learning Trajectories and what we learned.
- VII. Concluding Remarks: Why is the Cartesian Connection so Problematic?

Note: We will omit literature review and other background so we can focus on problems and the landscape within which the trajectories reside.

Background on the Students

- This course (Math 181AB) is for undergraduate math majors in the pre-secondary teaching track of the UCSB Math Major.

- In California undergraduates choose a major and then attend graduate school to obtain a teaching credential. In K-6 the credentials are “multiple subject” and in 7-12 they are “single subject” (of course certification in additional topics is possible, usually by taking a test.)

- The pre-service secondary math teachers in our courses are almost all math majors (although a math major is not a credentialing requirement.)

Course Objectives

- The course combines RME problem solving with case studies of K-8 classrooms, primarily Math in the City CDROM.
- We are interested on their combined influence on the abilities of the students to make sense of the work children do in related tasks.
- Today we focus on the relationship between the Function concept and Covariation. These are important ideas that in K-16 mathematics in number and operation, early algebra, number theory, the calculus, although this terminology may not be used in all those domains in exactly the way I use it.

II. Guiding Assumptions, Features of the Course, and the Research Question

• **“Kidwatching” is a critical first step**.
“Kidwatching” is a critical first step in the education of teaching professionals because everything we do has to be grounded in our understanding of the development of the learner.

• **Mathematizing new Contexts.**
At the same time, teaching professionals need to consistently mathematize new contexts and construct new understandings because it heightens their abilities to look for similar development in learners.

• **Reconstruction of K-12 Mathematical Big Ideas and Strategies.**
Further, we need to constantly reconstruct basic mathematics, seeking further connections. This can occur in tandem with kidwatching, bouncing back and forth between our understandings and the understandings of the learner.

- **Alternating Between Mathematizing and Kid Watching.**

In my instructional design I have my students alternate between mathematizing and kid watching, where if all goes well, my choices of contexts and classroom episodes are linked and will foster their seeing connections between the mathematics they are learning and the mathematics the children are learning.

- **Course Format:** There are two courses, each for one ten-week quarter, and each course is broken into three parts that focus on a particular topic, within which students alternately investigate RME contexts and then study children in grades 5-8 at work on RME type contexts.

• **Inquiry in the Course**. Since the students in these courses are mostly senior mathematics majors preparing for secondary teaching the inquiry we design for them is set at a collegiate level and consequently the contexts we design for them involve substantially more mathematical abstraction than you would expect to find in K-12 tasks. None the less we strive for the same features you might seek in designing RME contexts in K-12, in particular the context needs to be familiar (real or imagined), require sense making and mathematizing to get started, and then require construction and use of important mathematical ideas to create a solution. Often we turn to problems motivated by examination of the history of mathematics as is the case the two examples we consider today.

• **Case Studies.** Most of the children's work examined come from the CDROM developed by Maarten Dolk and Catherine Fosnot at Math in the City at CCNY and the algebra case studies of Smith, Silver, and Stein (2005). These are selected to raise mathematical ideas we identify as closely related to those arising in the inquiry of the course.

Research Questions

- Are adult learner's strategies linked to how they interpret children's work, and if so how?

- Are adult understandings of big ideas linked to how they interpret children's work, and if so how?

- How does this inform our work with pre-service teachers, and what might this guide is in designing learning trajectories for courses for prospective teachers?

III. Functions and Covariation, what are we after here?

Consider the arithmetic sequence 3, 9, 15, 21, 27,…

When asked to describe this sequence, children may say "you add six each time, starting with three".

This is *covariation*. From our point of view they are comparing the two sequences

0	1	2	3	…	as this goes up by 1
3	9	15	21	…	this one goes up by 6

Later in the curriculum, it might be encoded as the values of the function $f:\mathbf{N} \rightarrow \mathbf{N}$ given by $f(x) = 6x+3$. This is viewing the sequence as a *function*.

Eventually, we learn to hold both of these concepts in our mind simultaneously and we want our students to learn to do so too.

More formally, one can give the following definitions.

When discussing function we will consider functions $y = f(x)$ of one variable x with specified domains and ranges, e.g. we take the Dirichlet (or correspondence) notion as a concept definition.

When discussing covariation we will consider a pair of functions $y = f(x)$ and $z = g(x)$, where formally covariation describes a relationship between y and z that holds for all (or specified values of) x.

In the case of the arithmetic sequence 3, 9, 15, 21, 27,…, to describe the covariation noted by the child, we are really comparing the two functions $y = \mathrm{f}(x) = 6x + 3$ and $\mathrm{z} = \mathrm{f}(x+1) = 6x + 9$ and the expression of covariation in this case would be $z = y+6$.

If you don't like this formalism, don't worry, we won't rely on it too much in this talk. But these definitions allow us to see how pervasive these ideas are throughout mathematics as we illustrate next.

Other examples of functions and covariation

- Covariation in part-whole relations: if $P_1+P_2=W$, then $(P_1+x)+P_2=W+x$.

- Repeated addition for multiplication is covariation: 3 + 3 = 6, 6 + 3 = 9, 9 + 3 = 12, so 4 x 3 = 12.

- If f(*x*) = *mx* + *b,* then the slope *m* an be understood as a measure of covariation.

- An ordinary differential equation, say *y*' = *y*, is an expression of covariation between the two functions *y* = f(*x*) and $\underline{y}$ = f'($\underline{x}$).

- Euler's method and other iterative methods for solving Differential equations are based on covariation, while they converge to describe a function. (These are based on discrete versions of the ODE.)

Opening Investigation from the Course

I have used the following investigation over a number of years with undergraduate math majors headed towards teaching. It provides a setting for discussions of proportional reasoning, language, and the development of mathematics, both historically and for the learner. As you will see the idea of covariation is central.

Math 181A Investigation

Given below are some results stated using mathematical language that is close to that used by those who discovered the results (of course these mathematicians didn't use English!) Your task is to make sense of what they mean and to translate these results into modern notation. (You can check to make sure your translation is correct by proving the result!)

1. Translate the following results of Eudoxus (410-355 B.C.E.) into modern terminology:

(i) "Circles are to each other as the squares on their diameters."

(ii) "Spheres are to each other in the triplicate ratio of their diameters."

(iii) "A pyramid is the third part of a prism having the same base and the same height."

2. (Thymaridas, 4th Century B.C.E.) "If the sum of quantities be given, and also the sum of every pair which contains a particular one of them, then this particular quantity is equal to the difference between the sums of these pairs and the first given sum divided by number of quantities less two." (This is stated in so-called "rhetorical algebra".)

The students usually spend a full 90 minutes on this investigation the opening of the course. The first question can eat up the entire time for some of the students.

Take a few minutes and debate with your neighbor what you understand as the mathematical content of Eudoxus' statement:

"Circles are to each other as the squares on their diameters."

You may have recognized this result!

It is Proposition 2 from Book XII of Euclid’s Elements.

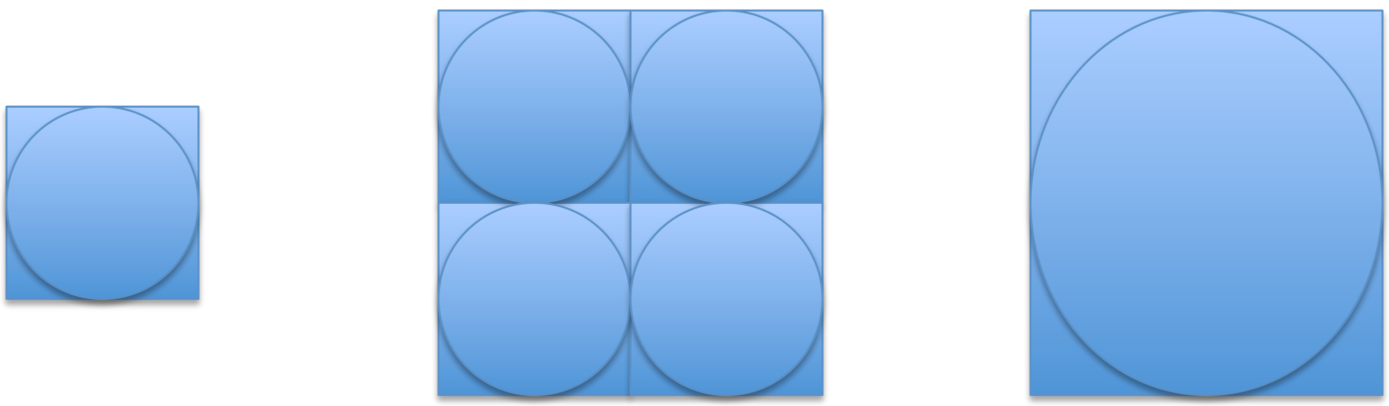
So we should expect prospective high school teachers to understand what it means even if they are not (initially) used to the language in which it is presented.

It is actually a statement of covariation.

So what do my students do with this statement?

- Many Students draw circles and draw a square on the diameter and believe the statement is about the figure.

- Some students draw a square grid and place circles inside it and assert he is saying that the circle fit inside the squares. They note that if they double the diameter, then they can put four circles inside the larger square. This gets at covariation (as a geometric model) but is limited to integral scale factors.



- The students who think about area assert the Eudoxus statement means:

$$\text{Area} = \pi r^2 = \tfrac{1}{4}\pi d^2 .$$

- They then check that if you double the radius of a circle you quadruple the area, and so forth. Most believe Eudoxus has stated the area formula and miss the covariation. This is robust and persists into the papers they write. This is the phenomenon that motivates the trajectory we look at todayl.

- Some students assert that it is that the ratio of the area of two squares is the same as the ratio of the areas of the circles. Ultimately they derive an expression $A_1/A_2=d_1^2 /d_2^2$ (but they rarely start with it.)

- Other students assert that if you scale the size of a circle, then its it area increases by the square of the scale factor. Something like “The scale factor squared determines how much bigger the area of the second circle is from their initial one.” This typically emerges later and at times with substantial intervention.

After class discussion, posters are presented. Here is an example.

“Circles are to each other as squares on their diameters” (Eudoxus)

The scale factor from one diameter to another determines the difference in area. Specifically, the scale factor squared determines how much bigger the area of the second circle is from the initial circle.

Area = πr^2
➔ $\pi(2r)^2$ ➔ $4\pi r^2$
➔ $\pi(3r)^2$ ➔ $9\pi r^2$

In the class discussion, it becomes apparent that the majority of the students are not aware of the fact that the number π plays no role in the result Eudoxus describes. Even after class discussion about this, they will later write in their papers “Eudoxus statement proved the formula that the area of a circle is πr^2 .”

I leave this problem as an opener for the class because it also raises questions of language, the nature of mathematical discovery, the importance of representation, and other issues I wish to introduce. But as far as covariation is concerned, my work is cut out for me!

The research describe next attempts to address the following.

(1) How can we be reasonably sure that students really understand the covariation and function in contexts that are relevant to K-12 teaching, and

(2) What can we say about learning trajectories that indicate success in building understanding of this relationship.

IV. Student work on an assessment at the close of the instructional sequence.

In this section the assessment given at the close of the unit is described. It is a written paper and they had a week to complete it. The assessment measures:

(1) The ability of students to interpret covariation in a position vs. time context and solve a problem involving the two functions (Frank and Doris' Race). The problem is reminiscent of problems give in the 1990's "Calculus Reform".

(2) The ability of students to recognize the role of covariation in a pair of fourth grade students (AnaMaria and Kevin) considering the distinction between the two forms of division. Previously the class had studies children working on this problem, including this pair, and they were provided with a full transcript of their discussion and a copy of their written work.

As you will see shortly, prior to completing this assessment, the students were explicitly introduced to the notions of covariation and function in multiple contexts.

Math 181A Paper: Due January 26, 2009

In this paper you are to write about the relationship between covariation and functions, and two instances where this relationship can be represented with the area model. You are to solve the problem, Frank and Doris' race, and you are to compare your work on this problem with the class' discussion of the work of AnaMaria and Kevin on the Soda Machine problem. This at first may seem like quite a leap: your work on Frank and Doris's race implicitly uses the fundamental theorem of calculus, while AnaMaria and Kevin are grappling with the distinction between partative and quotative division. But the cognitive issues are related, and you need to make sense of and describe this relationship.

Your paper should include:

- 1. A solution to the problem about Frank and Doris' race. Identify the role of covariation and function in this problem you should explain the meaning of "area" in the graph provided with this problem, as well as

- 2. A discussion the transcript of the class discussion of work of AnaMaria and Kevin on the soda machine investigation and an explanation of the mathematical issue the students are grappling with. Identify the role of "area" in the representations of the problem context. While reading the transcript, keep in mind that Tanya is the teacher. We can arrange for you to view this discussion again if you like.

3. An explanation of how the cognitive issues involving covariation and function are related to the issues confronting Tanya's class. To help you, think about the function $f(x) = 6x$. Note that $f(n+1) = f(n) + 6$, and $f(x) = x+x+x+x+x+x$ are two ways to understand this function. The area model and Cavalieri's Principle should help you too!

This paper is not to exceed five pages (diagrams not included.)

We next take a close look at Frank and Doris' Race, and the work of AnaMaria and Kevin on Tanya's problem. We will also consider the rubrics used to evaluate the students' work.

Frank and Doris' Race

Frank and Doris are good friends who both love drag racing (both are engineers). They each bought cars and are busy preparing them for the July 4 Shelbyville race. Both cars are similar, but Frank has installed a small nitrous tank to give him an extra boost towards the end of the race. But it also slows him down earlier on. Below is a speed vs time chart for both cars obtained from their theoretical models:

Time (seconds)	Doris' Speed (ft/sec)	Frank's Speed (ft/sec)
0	0	0
5	50	20
10	85	85
15	120	140
20	130	150
25	140	155
30	150	160

If these model's predictions are accurate, who will win the quarter mile race? When will Frank pass Doris?

Discussion of Part 1. If you look at the covariation that relates Doris' speed to Frank's speed you will see that Doris starts off with a lead, Frank matches her speed at ten seconds, and them it appears he will be able to pass her at some point down the track. Exactly where will require some estimates of distance traveled (numerical integration or graphing and estimating area under the curve), but should be able to estimate reasonably well who will reach 1320 ft first and perhaps where Frank passes Doris. Both speed and distance are functions of time here. The cognitive obstacle one looks for in this type of problem is students believing that Frank passes Doris at 10 seconds when their speed matches.

In analyzing the undergraduates work on Frank and Doris' Race the following rubric was used.

Level 1: Unable to solve the question or use position function formula from physics or mathematics class.

Level 2: Solve the problem, but unable to see the distance function is developed from covariation concept in their work, i.e., step by step accumulation of areas under the curve.

Level 3: Solve the problem and see function is developed from covariation concept in their work, i.e., step by step accumulation of areas under the curve.

The papers were also coded to obtain further information about the strategies used:
O: Obstacle in interpreting speed function, i.e., 10 seconds Frank overtakes Doris.
M: Uses distance as area under the curve.
A: Use average speed to find the area under the curve for 5-second interval period.

Discussion of Part 2. The student work of AnaMaria and Kevin is from exploring Soda Machines (Dolk&Fosnot, 2005). The children have been asked to solve two problems concerning two soda machines:

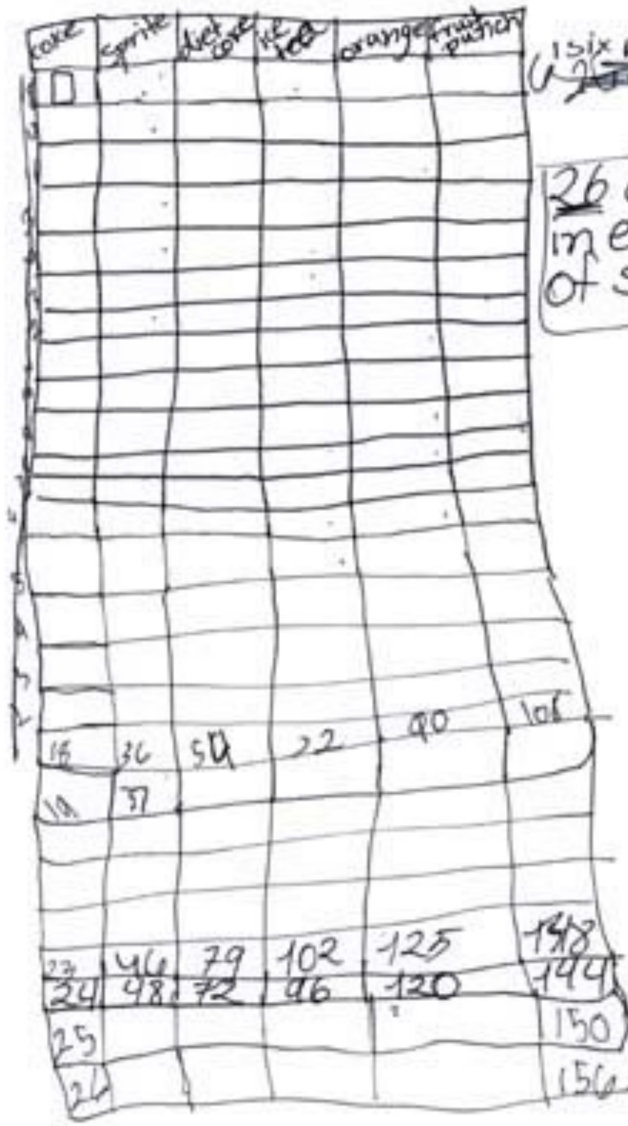
(1) If a soda machine contains 156 cokes, how may six-packs would that be?, and,

(2) If a soda machine has six flavors and 156 sodas, how many of each flavor would that be?

These two questions model the two forms of division.

Lets watch AnaMaria and Kevin work on this problem and then we will look at their poster.

Ana Maria & Kevin



1 six pack
~~26 can fit~~

26 can fit in each row of soda.

26 6 packs

Note that AnaMaria are working on the second problem, but in their they have found (in our notation) that if $f(x)=6x$ then $f(x+1)=f(x)+6$. This is one of the forms of covariation studied in the course. And for AnaMaria and Kevin, the understanding of covariation provided the link between the two forms of division.

In analyzing the undergraduates work on AnaMaria the following rubric was used.

Level 1: Mentions covariation and function at abstract level in words or forms, such as covariation is …function is …, or $f(x+1)=f(x)+6$ for covariation and $f(x)=6x$ for function), but not within the context.

Level 2: Notices covariation as $f(x+1)=f(x)+6$ and function as $f(x)=6x$ within the context, but the connection is not clear.

Level 3: Notices covariation and function concept in AnaMaria's second problem and provides the relationship between them.

The papers were also coded to obtain further information about the the students understanding of covariation in the context.

U: Mentions unitizing 6 packs or 6 flavors to explain covariation or correspondence view.
D: Discusses children's understanding of division through multiplication.
P: Discusses the relationship between quotative and partitive division.
M: Discusses role of area model in the context.
R: Discussion on how AnaMaria and Kevin's work and their work on Frank and Doris are related in terms of covariation and function.

The next table summarizes the results of the evaluation of papers completed by two cohorts of Math 181A students, each following different curricular trajetectories. The first number indicates the rubric score for Frank and Doris' Race and the second AnaMaria and Kevin's work.

Assessment Outcomes

	1,1	**2,1**	**3,1**	**1,2**	**2,2**	**3,2**	**1,3**	**2,3**	**3,3**
Cohort 1 N = 22	5	4	1	0	0	0	0	7	5
Cohort 2 N = 15	4	5	0	2	2	0	0	1	1

Two aspects of this analysis are striking. The first is that in both groups, the ability of an undergraduate to differentiate the roles of covariation and function in their own work on the Frank and Doris problem correlates highly with their ability to find it in the work of children (this is indicated by the low number of scores in the center of the table.) The second is that the Cohort 1 undergraduates were substantially more successful than the Cohort 2 undergraduates. We discuss the two trajectories next.

V. Two Learning Trajectories and what we learned.

The first trajectory is from the second iteration of the course, and the second trajectory is from the third iteration of the course that has been funded by my CCLI grant. The investigations in the course had been taught in a similar course over the previous decade, but without the case studies.

First Trajectory: Blended both Discrete and Continuous contexts, along with case studies.

Second Trajectory: Moved the unit four weeks later into course and then used “collegiate” continuous along with case studies as in first trajectory.

- **First Trajectory: Blended both Discrete and Continuous Settings**

- Because the prior year students had difficulties in recognizing the covariation in the Eudoxus problem and because they had serious difficulties with another problem based upon Omar Khayyam's solution of cubic equations (discussed at the end of the talk), it was decided they needed opportunities to reconstruct the relationship between covariation and function in both discrete and continuous settings, and after interacting with these contexts to have discussions that result in explicit definitions of the concepts.

Cohort 1 Trajectory

- Investigation #1: Water Tank Problems (related rates, similar to Calculus)
- Homework: Reading Mathematics vs Mathematizing and Landscape of Learning Chapters 1 &2 from <u>Young Mathematicians at Work, Constructing Fractions, Decimals and Percents</u>, Fosnot&Dolk
- Homework: Fibbonacci Path counting Problem (C.D. Stinger)
- Class Discussion: Readings, Homework and Covariation - Function Defintions
- Investigation #2: Tile Pattern, What is the Story, Discuss $dy/dx = y$, $y(0)=1$, Indentify Covariation and Function in these contexts.
- Class Discussion: Covariation and Variation in Investigations #1 and #2.
- Class Discussion: Case of Ed Taylor
- Investigation #3: Triangular and Pentagonal Numbers
- Class Viewing and Discussion: The Soda Machine (Dolk Fosnot)
- Paper #1 (Assessment). This occurred week 5 of the quarter.

Second Trajectory:

Interviews with the students at the end of the second year iteration revealed that they felt that the lack of "challenging, university level problems" was discouraging to them. At the same time, as instructors, we were very concerned that they develop a more solid understanding of the basic representations used in middle school mathematics than either of the first two cohorts developed (ratio tables, double number lines, etc.) So the decision was made to introduce the basic representations at the opening of the course, and to address the covariation and function questions in the second (of three) units in the quarter.

- But instead of moving the prior learning trajectory forward, the contexts used in addressing the covariation question were almost exclusively continuous, and there were three substantial investigations where the distinction was raised. A number of examples in the discrete setting were considered in class discussion, but the time spent on inquiry was what one might consider as "calculus based". This was an attempt to draw the students into the inquiry more deeply and at the same time to devote even more time to the topic.

Second Trajectory Introductory Unit

• Investigation: Sorting Numbers by Number of Factors
• Homework: Reading Chapters 1 & 2 from Fosnot-Dolk,
• Reading and Class Discussion: Strategies, Big Ideas and Models from developing
Early Number Sense.
• Class Viewing and Discussion: Sharing Submarine Sandwiches(Dolk Fosnot)
• Homework: Egyptian Fractions and Fuel Gage Problem (double number line involving fractions)
• Investigation: Frog and Toad Jumping and LCM(n,m)GCF(n,m) = nm (double number line involving multiplicative relationships of 8 and 12.)
• Homework: Fraction Minilesson CD and work with open double number line
• Investigation: A 6-Pack Packing Problem
• Paper #1 on The Role of Representation (both in collegiate contexts and in the cases viewed so far.)

Second Trajectory: Unit on Covariation and Functions

- Homework: Figurate Numbers
- Investigation: Inflation to Deflation.
- Investigation: Water Tank Problems.
- Class Discussion: Case of Ed Taylor
- Homework: Proof by Tomatoes , but linked to an integral formula,
- Investigation: Back to Basic ODE's
- Class Viewing and Discussion: The Soda Machine (Dolk Fosnot)
- Paper #2 (Above Assessment). This occurred week 7 of the quarter.

We note that each of the three investigations are "Calculus based", both in the nature of the questions and in the availability of the tools of calculus as a solution.

Again we recall the outcomes of the assessment.

	1,1	2,1	3,1	1,2	2,2	3,2	1,3	2,3	3,3
Cohort 1 N = 22	5	4	1	0	0	0	0	7	5
Cohort 2 N = 15	4	5	0	2	2	0	0	1	1

So again we note the distinction between the two groups performance on the final assessment. *It was really quite a surprise,* as we expected the second group to have made better sense of the student work and understand the role of covariation and function in Frank and Doris' race than the previous cohort. This was not the case. Interestingly, Cohort 2 also proved to be more adept at the use of the Calculus in its problem solving. But that did not translate into sense making about covariation and function and the work of AnaMaria and Kevin.

VII. Concluding Remarks, and Why is the Cartesian Connection so Problematic?

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- Undergraduate math majors preparing for secondary teaching have difficulty understanding the distinction between the expression of covariation in Eudoxus assertion that circular area grows according to the square of the diameter from the functional expression Area = πr^2 .

- The abilities of these undergraduates distinguish between covariation and function in speed vs time context at an early Calculus level appears to correlate with their ability to make sense of children's work on a division context where they are thinking about the two forms of division.

- Of the two curricular trajectories, the trajectory that had these undergraduate math majors investigate multiple contexts linking covariation and function in calculus contexts (related rates, contexts involving inflation and approximation methods in ordinary differential equations) were significantly less successful than a trajectory that used discrete analogues, tile patterns and figurate number problems along side related rate problems. The second group included strong students who overall received slightly higher course grades (3.4 vs 3.2) and who were significantly more successful with the calculus based tasks.

This occurred in spite of the fact that students repeatedly expressed a desire for the course to emphasize "calculus and beyond college level problem solving".

The research presented here, as well as other work we have done comparing this group of undergraduates with undergraduates in a course for pre-service elementary teachers, indicates that the typical mathematics major preparation is not as useful as many claim for middle and secondary school teaching. Students successful with both years of the calculus, abstract algebra, and analysis, can miss the development of mathematical ideas in middle grade students, while others, who reconstruct these ideas during their collegiate years appear to be more successful.

A Closing Example.

To close I would like to report on the work of these same two groups of students the subsequent quarter on an investigation that involves the *Cartesian connection*.

For the purposed of this discussion, a student understands the Cartesian Connection if they can effectively use their understanding that the that a graph of an equation in the x-y plane is a locus of points, and don't behave as if it is object "defined by the instructions" given by the ingredient symbols of the equation. You will see more what this means as we consider the example.

Math 181 B **Omar Khayyam and the Cubic**

Warm-up Question: What does it mean to "solve an equation"? Try to describe several perspectives. The quadratic formula enables one to "solve" any quadratic. What information does it give? How useful is it? For the ancient Greeks, solving an equation meant "finding a length with the desired properties." So for example, they could solve the equation $x^2 = 2$. How? Does the quadratic formula give you such a geometric solution?

What Omar Did: In addition to discovering Pascal's triangle over 500 years before Pascal, the Persian poet and mathematician Omar Khayyam (1050-1130) solved the cubic equation $x^3+(p^2)x= p^2q$ where p and q are natural numbers by looking at the intersection of the parabola $py= x^2$ and the circle $x^2 + y^2 = qx$.

1. Your task is to figure out how this intersection does lead to a solution of the original cubic.

2. Write out Omar's discovery in the language that he would have used. (This is the reverse of what we did in the mathematical language project.)

What Happens With this Investigation.

After the warm-up discussion (typically half an hour with digressions dealing with the historical and pedagogical importance of the measurement model of number), a problem immediately arises in graphing the circle $x^2 + y^2 = qx$.

Who teaches high school here and can tell us what happens?

The students graph parabola adequately, but graph the circle centered at the origin with radius labeled . This illustrates student use of the equation as a set of instructions to describe a circle, instead of thinking carefully about the circle as a locus of points.

But is this a silly glitch that undergraduates will quickly recover from, or is it more serious?

The intervention is the obvious one, after students have the parabola and circle drawn they are asked to consider why the point (0,0) is a solution of both the parabola and circle's equation, but does not appear as an intersection point. Fast forwarding, over one full hour after this intervention we have the following presentation to the class.

We next take a brief look at the math congress on this investigation in the earlier Cohort 1 class during the second quarter of the course. The math congress began with presentations of three groups which all came up with circle(s) centered at the origin for the algebraic representation . Although the form has on the right side of the equation, without attending the presence of variable *x*, the three groups simply transformed into a circle (or circles) centered at the origin with radius . Then the fourth group, John's group, brought an interesting idea of "circle with varying radius".

Here is what John says:

John:	First we drew circle with radius length . And then we kind of noticed that as you go close to equals zero then radius length goes to zero also. So the circle wouldn't really make sense there and we ended up getting with this, like infinity looking symbol.

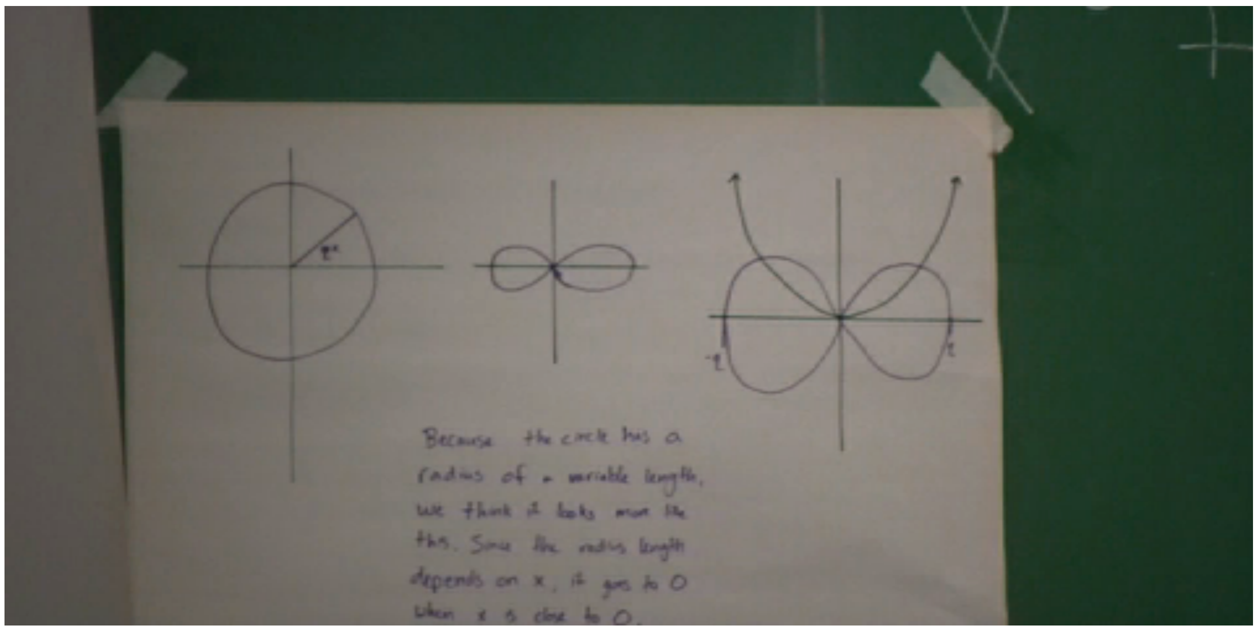

Figure 1: John's representations

Dr. B: So do you believe in this (the circle centered at the origin)? Do you believe in this (the bow-tie figure)? What do you believe in?

(Dan's group)

Dan: I am kind of leaning toward the second last kind of circle because if goes to zero then radius goes to zero. So it is going to be a point.

Dr. B: So if radius goes to zero, then it is not a circle, it is a point, not circle anymore, huh?

Dan: Exactly!

Dr. B: What do you mean radius goes to zero? It looks like it has radius in here (Dr. B points Figure 1).

Dan: I know, but if we set equals zero, then equals zero, so it is a point.

John: It could be circle changing the size of radius based on .

Dr. B: So what is changing?

John: The radius length.

Other students weigh in on John's Diagram and explanation.

Kara: We think it is bow-tie one. When we were working with the equation, we got zero zero ( ) as the solution.

Dr. B: Oh, so zero zero is a solution. So you think it might be something like this (the bow-tie figure).

Kara: Maybe it is only like the right of the bow-tie?

Dr. B: Only the right part. Why would you think like that?

Kara: Because he (Omar Khayyam) doesn't know what negatives are. Wait! You said he doesn't know negatives, right?

After about twenty minutes of similar discussion, the class has been convinced of John's ideas, except for one group that was instructed from the outset to keep quiet. At the close of the two-hour session, the group that completely nailed the problem presented.

Before leaving the students were asked to present a 5-minute quickwrite and answer the following:

(1) Give the definition of a circle, and
(2) Give a complete explanation why the equation, , defines a circle and explain the role of *a, b* and *r* in the process.

Most students could answer (1) without difficulty, but could not offer much, if any, explanation of what is going on in 2.

The second cohort of students we have been discussing today, had no difficulty completing the square and graphing the circle properly.

Both groups had algebraic calculations showing that the parabola and circle defined by Omar Khayyam did indeed provide x-values that are solutions to the cubic.

However, in both classes, most all students were not able to give a geometric definition of the x-value and therefore complete the task of determining how Omar Khayyam might have described the solution to the cubic in geometric terms.

We take the above as evidence that the students' understanding of the Cartesian connection is weak at best. We also noted that the ability of these same students to discuss the Cartesian Connection in collegiate contexts appears to correlate with their ability to identify similar issues in examining work students on tasks in graphing linear equations. Those students without a robust understanding typically identify student difficulties in graphing lines like, $y = 1$, $y = 2x$, $y = x+3$, $x = 2$, as "not remembering instructions" instead of referring to the Cartesian connection (I am over simplifying here).

We leave with the following questions:

- What instructional sequences do we need to design to help pre-service secondary teachers deepen their understanding of the Cartesian connection?
- Is there a link between students understanding of covariation and the Cartesian connection and how might we integrate these topics?
- Do we need to extend the framework developed by Tall, Vinner and others on the relationship between the concept definition and the concept image for function to covariation and the Cartesian connection, especially how the pre-service teachers' concept images influence how they interpret with their future student's thinking?

These are the questions that will interest me as we continue this type of work.