Helping students construct more formal mathematics

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Realistic Mathematics Education (RME)

- Freudenthal: teaching ready-made mathematics = an anti-didactical inversion; taking the end point of the activity of brilliant mathematicians as starting points for young children.
- “Mathematics as a Human Activity”
- An activity of solving problems, looking for problems & organizing subject matter
- Organizing subject matter of reality or mathematical matter
- Mathematizing; reinventing mathematics

“Realistic”

- Freudenthal: Reality is what one experiences as real; what is real for a mathematician is not the same as what is real for a layman.
- Starting points have to be experientially real = situations in which students can act and reason sensibly
- Learning mathematics as expanding reality, or as creating some new reality

- Reform math: construction instead of instruction

Realistic Mathematics Education

- How to use Symbols & Models?
  - Emergent Modeling
- How to help students invent what you want them to invent?
  - Hypothetical Learning Trajectory
- How to ensure student participation in problem oriented mathematics?
  - Classroom Culture/Social Norms
- How to foster the development of sophisticated mathematics?
  - Generalizing & formalizing

How to use Symbols & Models?

Symbols do not come with a given meaning!

⇔ Problems with the common view on learning

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A 'layman's view on instruction

- How do people learn?
- Layman's view: By making connections between what is known and what has to be learned

- How do people learn mathematics?

→Learning Mathematics: making connections with an abstract, formal body of knowledge

- Problem: Gap between the knowledge of the students and the abstract, formal body of knowledge

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What makes mathematics so difficult?

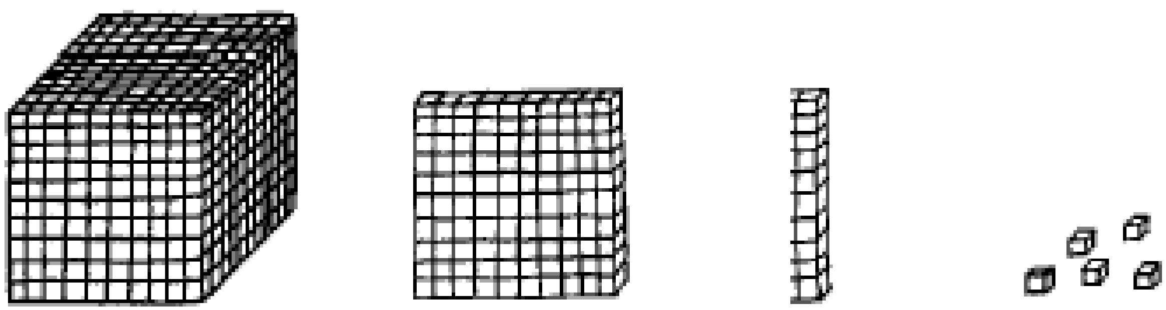
- The problem is not in the abstract character of mathematics as such
- It is the gap between the abstract knowledge of the teachers and the experiential knowledge of the students
  - Teachers and textbook authors tend (mis)take their own more abstract mathematical knowledge for an objective body of knowledge with which the students can make connections
- Students cannot make connections with knowledge that is not there for them

Problems with the common view on learning

- 1: The new mathematical knowledge the students have to connect with, does not yet exist for them.
- 2: The learning paradox
  - The symbols that one needs to get get access to a new mathematical domain, derive their meaning from that very domain.

Concretizing does not help

- Dienes Blocks: Trying to <u>show</u> the mathematics



- But how are the students to <u>see</u> the mathematics they do not know yet?

- E.g. for us one ten is at the same time ten ones; for young students these are two different things

The new mathematical knowledge does not exist yet

- Young children don’t understand the question: “How much is 4+4?

  Even though they know that “4 apples and 4 apples makes 8 apples”

- Ground level (Van Hiele):

  Number tied to countable objects: “four apples”
- Higher level: 4 is associated with number relations:

  4 = 2+2 = 3+1 = 5-1 = 8:2

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Miscommunication between teacher and students

- Student are thinking at the level of countable objects
- Instruction on the level of number relations;
  - Note: Telling students that 2+2=4, etcetera, will not help if the students do not know what ‘2+2’ means.

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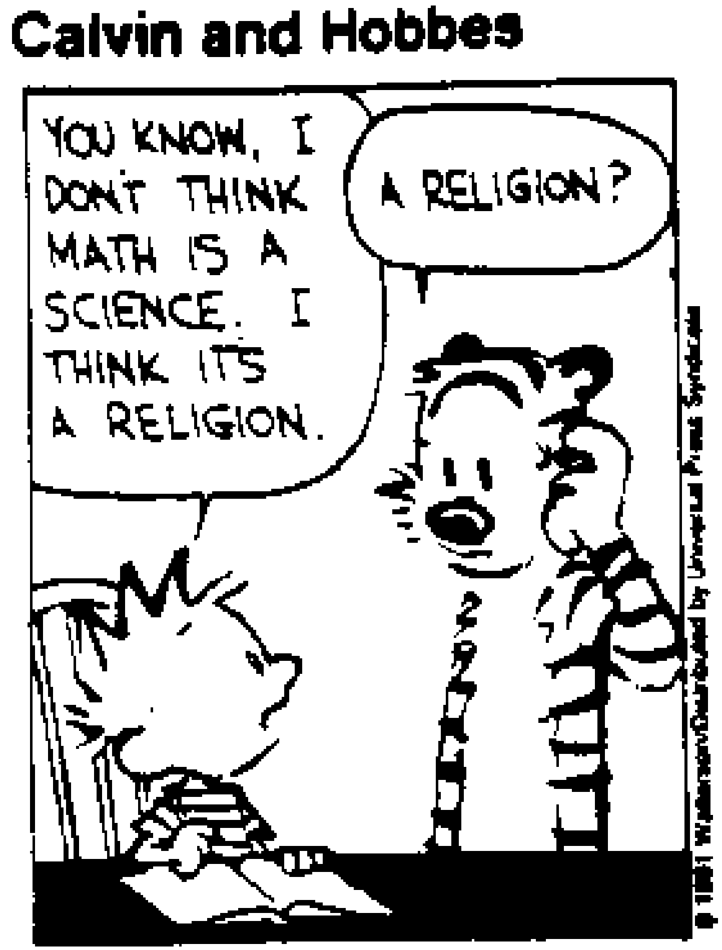
Consequences of the common view (connecting …)

- The body of knowledge only exist in the minds of teachers and textbook authors; how can students connect to a body of knowledge that does not exist for them?
- The learning paradox: Mathematical symbols derive their meaning from a certain mathematical domain. However, you need to understand those symbols to enter that domain.

Math is a religion



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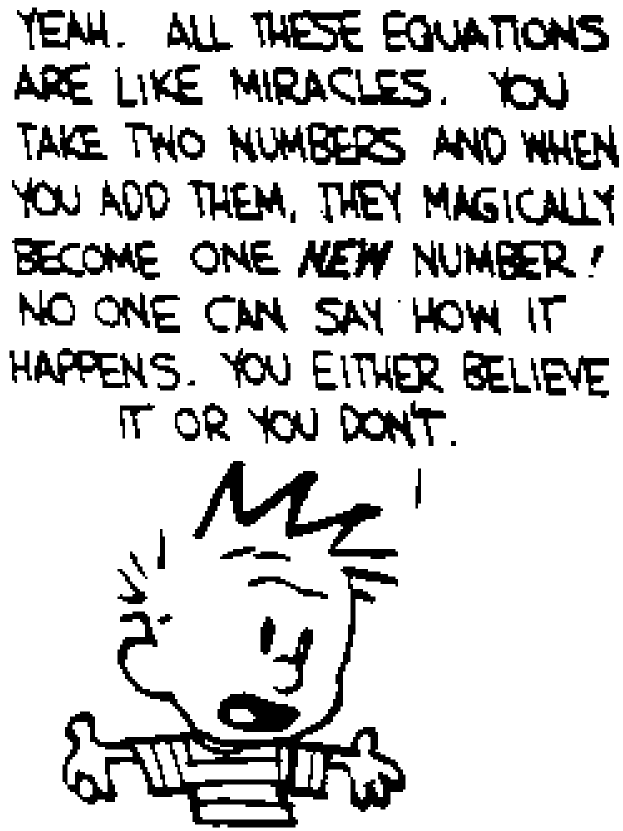
Math is a religion



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Math is a religion

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Alternative

- Symbols and representations emerged in history, starting with informal representations and intuitive understanding → symbols and meaning coevolved (Meira; Latour)
- Reflexive relation:
  - new symbols allow for new meaning to develop
  - new meaning can be expressed with new (more sophisticated) symbols
  - …

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RME: guided reinvention, emergent modeling

A statistics course as an example

- Suppose we choose 100 men at random, and line them up from small to tall.
- Make a drawing of what the line up would look like

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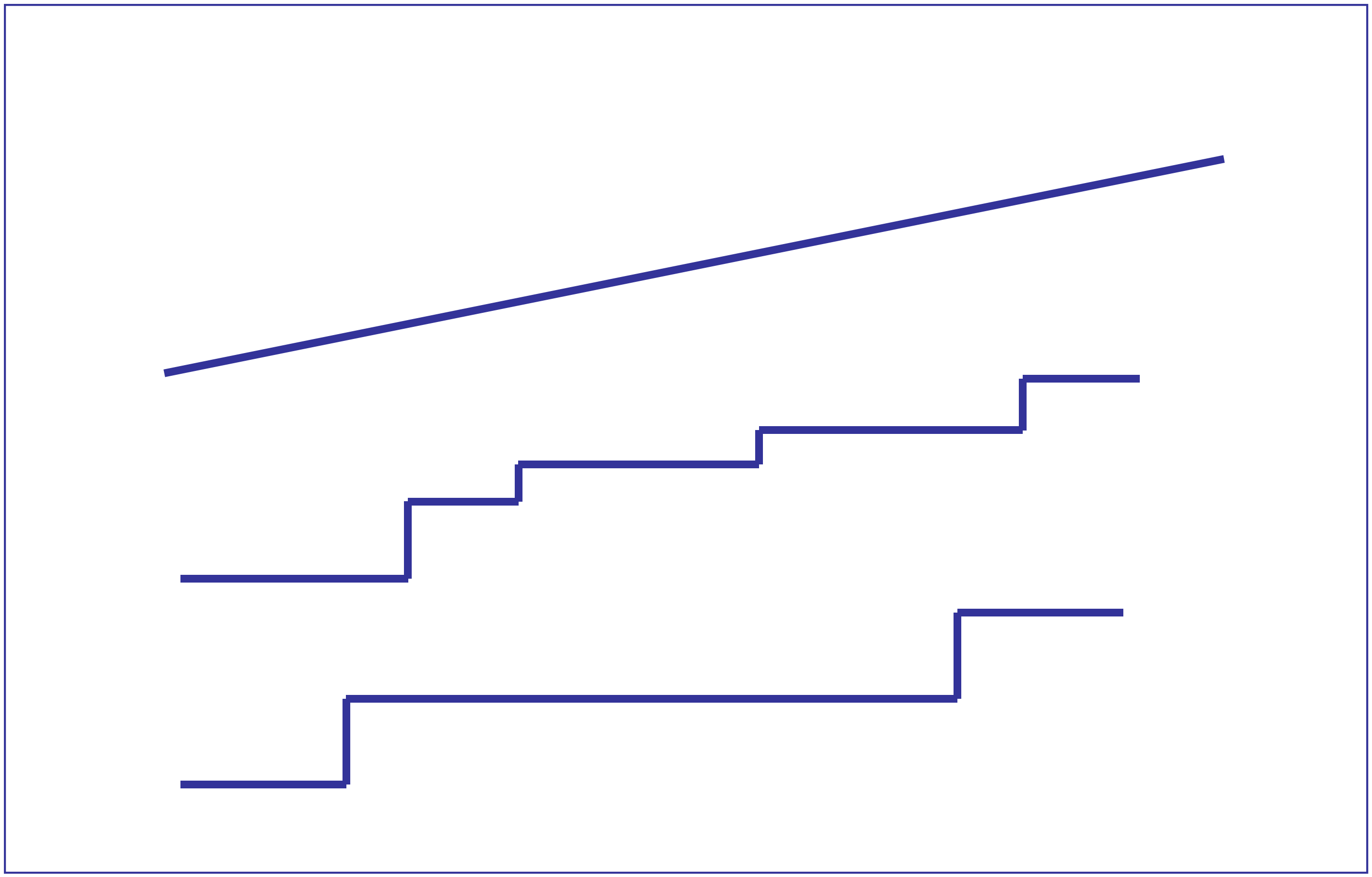

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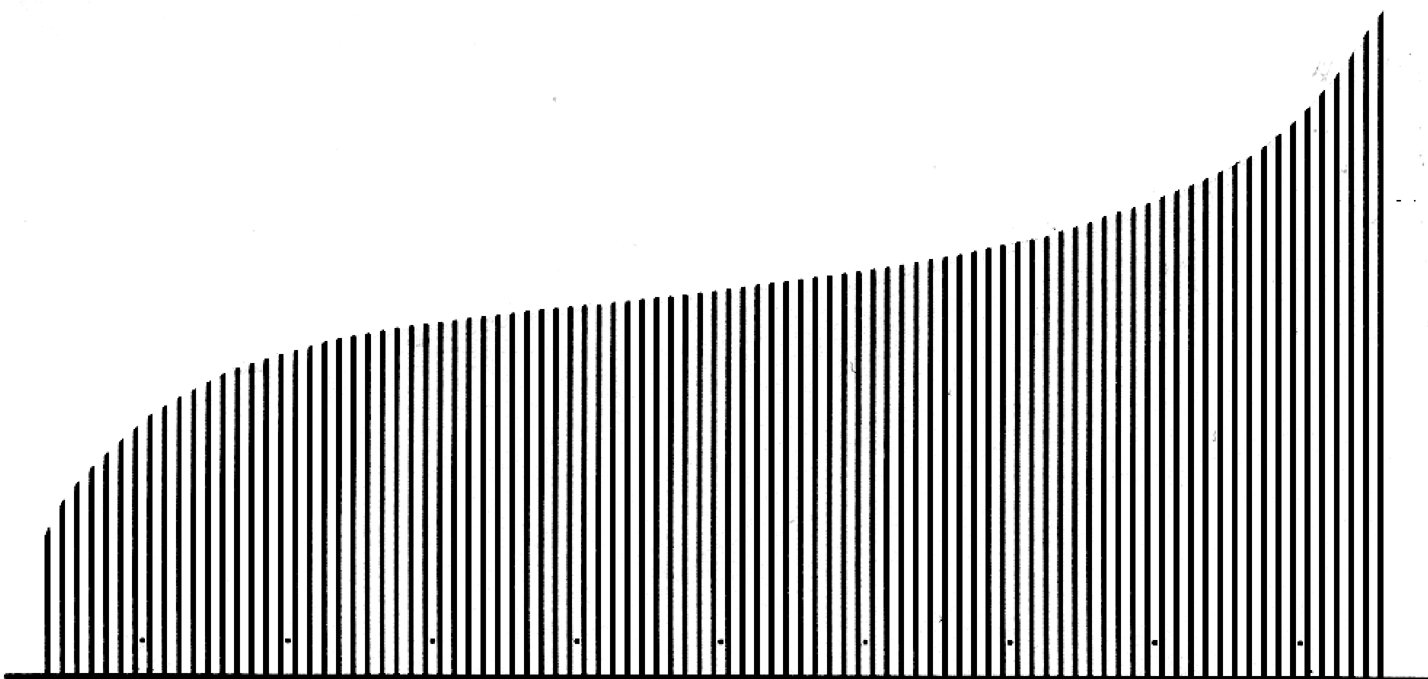
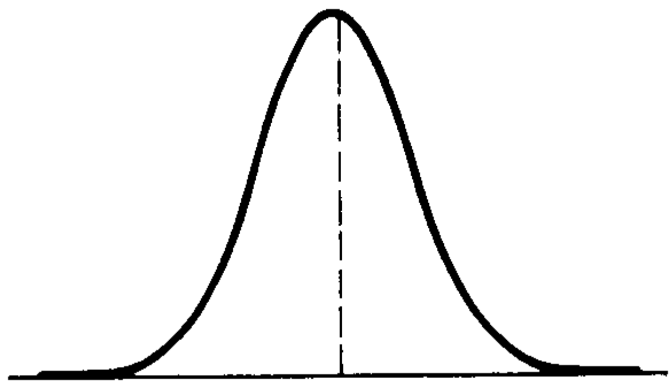


Distribution

- Suppose this graph represents the distribution of the heights of men.



- What does an arbitrary point on this curve signify?

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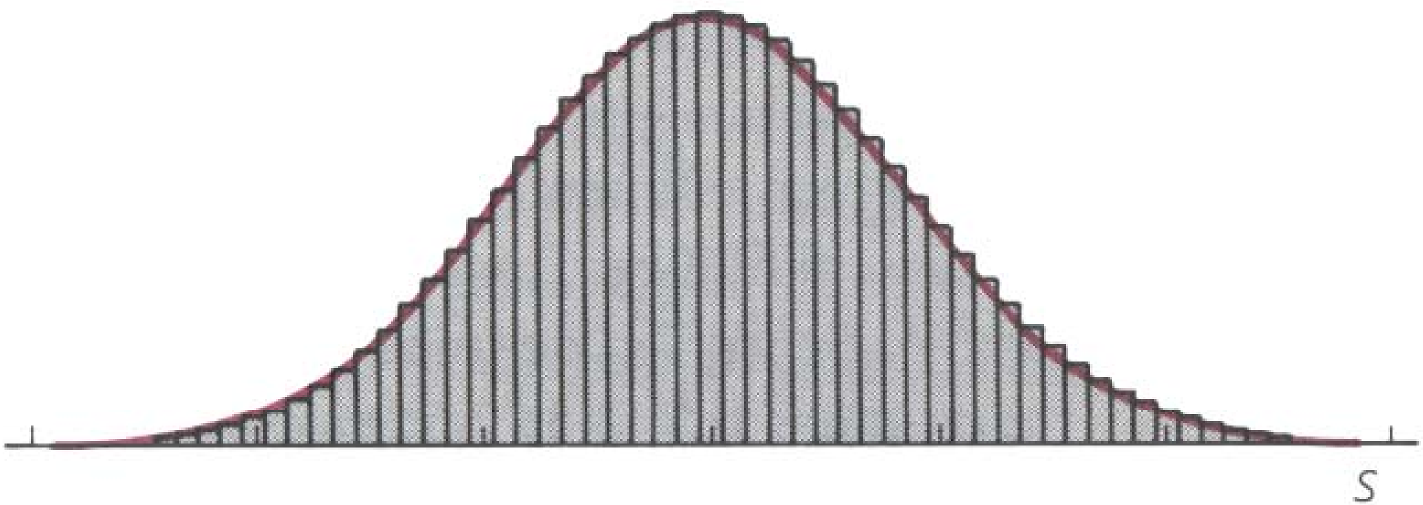
limit histogram

$\Delta \to 0$



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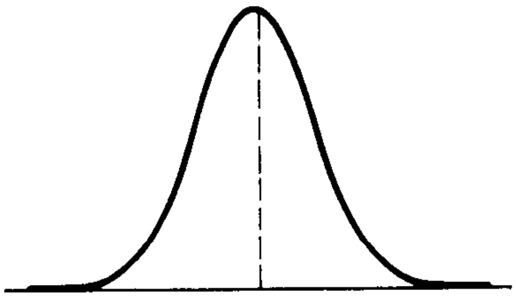
Distribution as a density function

- Height = density of data points around that value
- Distribution can be thought of in terms of shape and density
- Instructional goal:

  distribution as an object



mean, mode, median … as characteristics of a distribution

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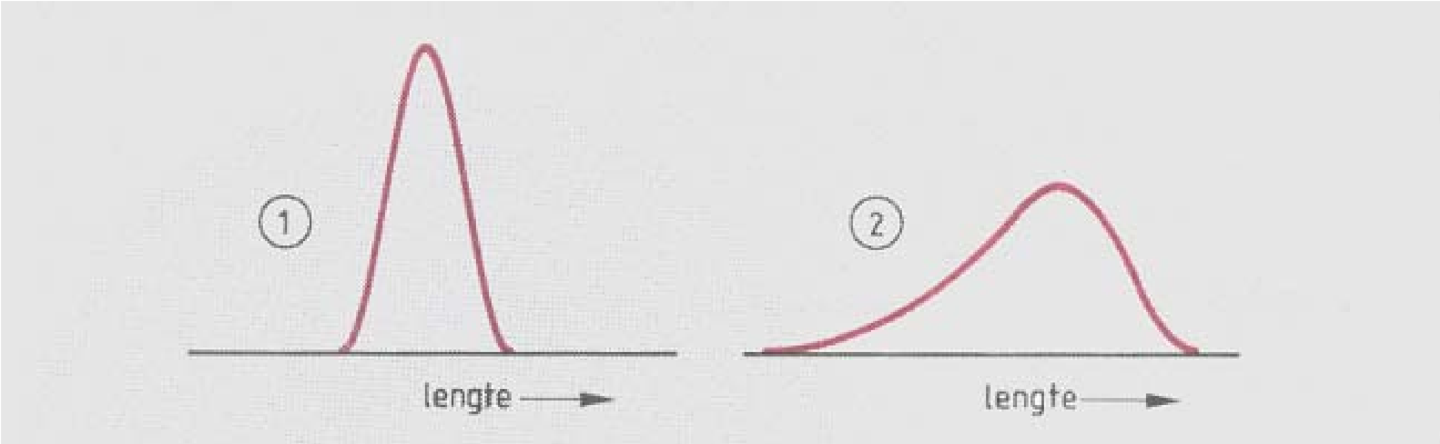
Distribution as an object



Characteristics:

- Position
- Spread
- Skewedness

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RME Heuristics: 1. Guided reinvention through progressive mathematizing

- Foster reinvention of tools & measures of central tendency (median, quartiles etc.) as means for getting a handle on a distribution => characteristics of a distribution

- Looking for starting points that are experientially real
  - Data creation (⇔ reason)
  - Informal graphical representations

RME Heuristics: 2. Didactical Phenomenology

- Present-day applications: starting points
  problem situations that may give rise to situation-specific solution procedures

- *phenomenology of mathematics:* how the "thought thing" (nooumenon) organizes the "phenomenon."

Didactical phenomenology

Sequence of “thought things”:

1. **Data points** as a means of organizing ***measurement values***

2. **Density** as a means of organizing how ***data points*** are distributed

3. **Distribution** as a means of organizing ***density***

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RME Heuristics 3: Emergent modeling

- Modeling as a student activity in service of a learning process

- Modeling activity in an experientially real, commonsensical setting
- Shift in attention – from contextual setting towards mathematical relations
- Creation of some new mathematical reality →

  a *model of* informal mathematical activity becomes a *model for* more formal mathematical reasoning

Example:
Teaching experiment with Paul & Cobb Erna Yackel

Modeling in the context of Data Analysis

- Starting point linear type of measurements & informal graphs:
- individual data values inscribed as bars (value bars)

  e.g.
  - lifespan of batteries
  - speeds of cars

Batteries

- Talking through the process of data creation:

  Consumer Report on batteries; what should be in it?
  - Price
  - Life span

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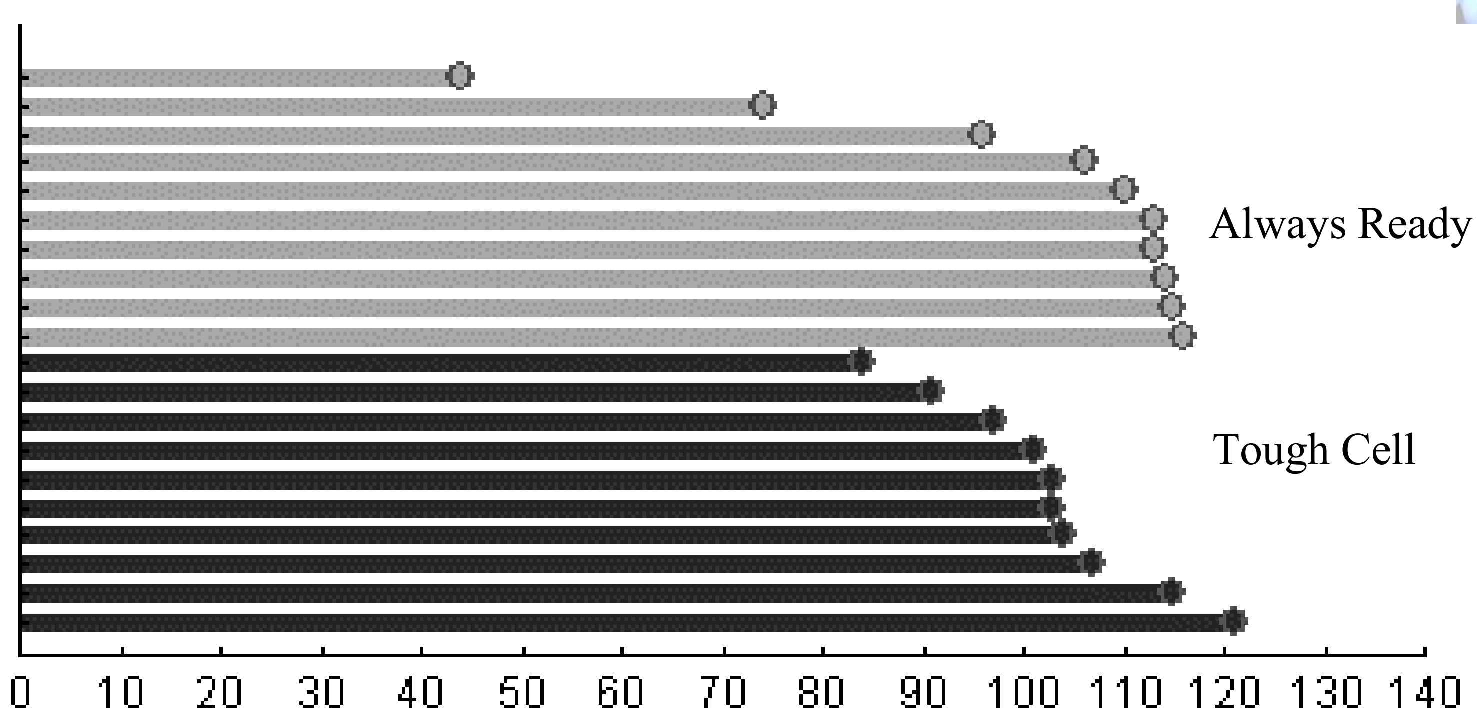
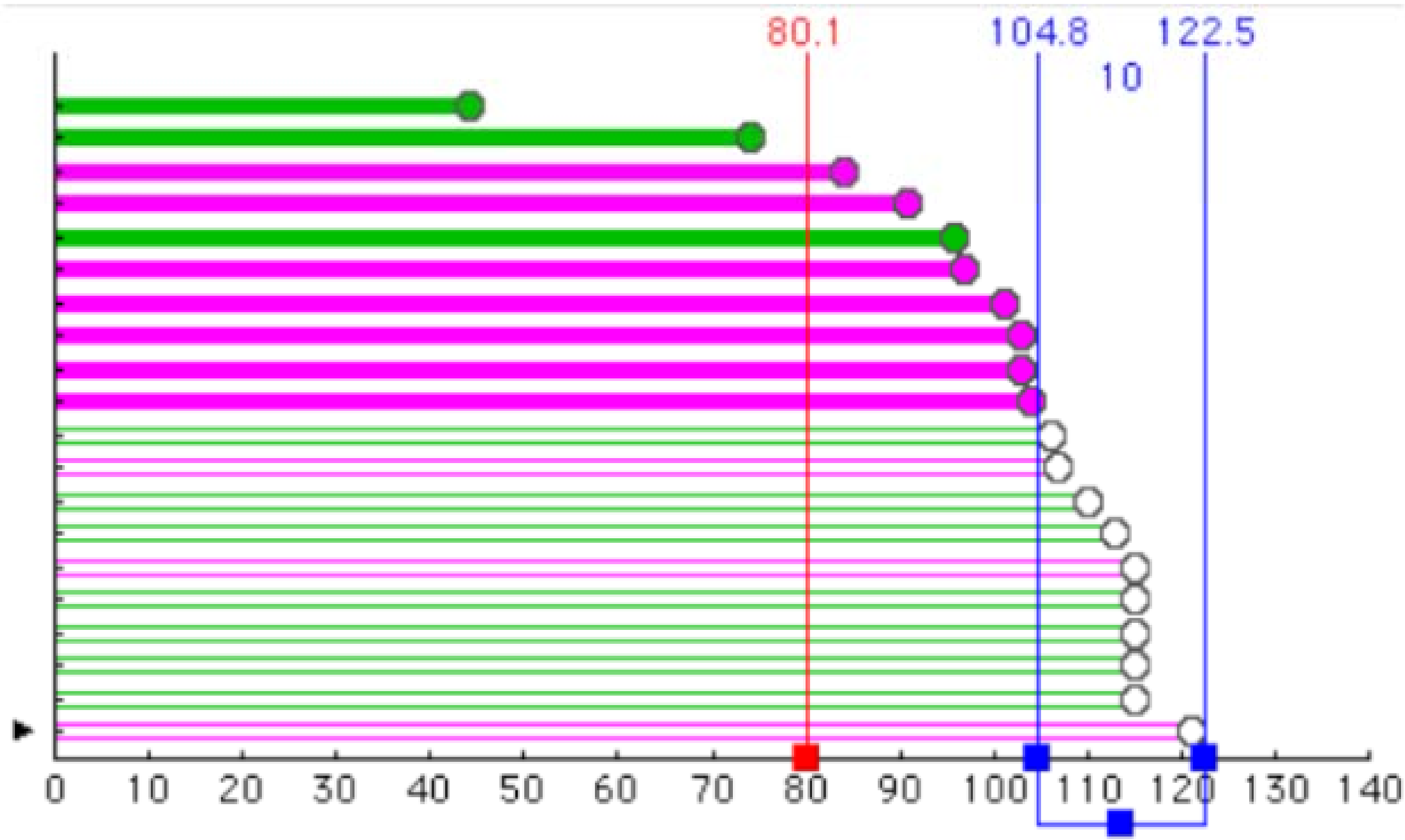
Battery Life Span



Classroom episodes
Battery life span



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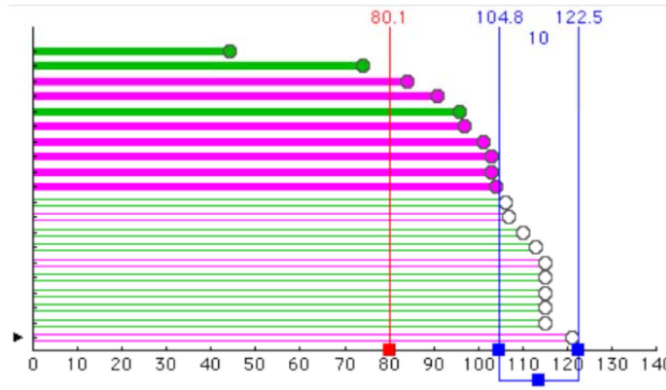
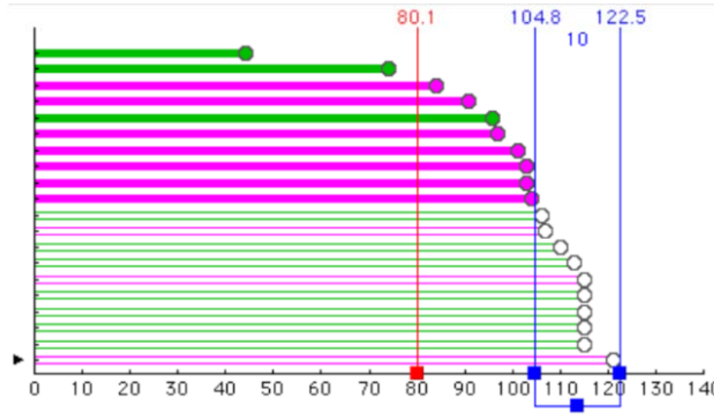


- Casey: And I was saying, see like there's seven green that last longer.

Janice: She's saying that out of ten of the batteries that lasted the longest, seven of them are green, and that's the most number, so the Always Ready batteries are better because more of those batteries lasted longer.

Teacher: Why "ten"

The next student to explain his reasoning, Brad, directed the teacher to place the value tool at 80.



- Brad: See, there's still green ones [Always Ready] behind 80, but all of the Tough Cell is above 80. I would rather have a consistent battery that I know will get me over 80 hours than one that you just try to guess.
- Teacher: Why were you picking 80?
- Brad: Because most of the Tough Cell batteries are all over 80.

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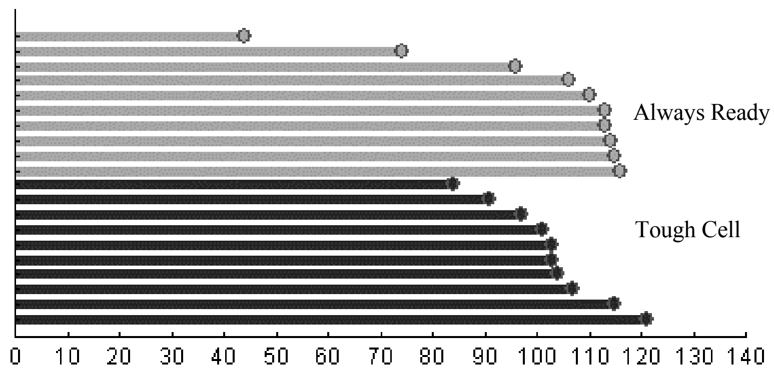
- Barry: Like, if you’re using them for something real important and you’re only going to have like one or two batteries, then I think you need to go with the most constant thing. But if you’re going like, "Oh well, I just have a lot of batteries here to use," then you need to have most of the highest.

- “consistency”

  ToughCell is more consistent



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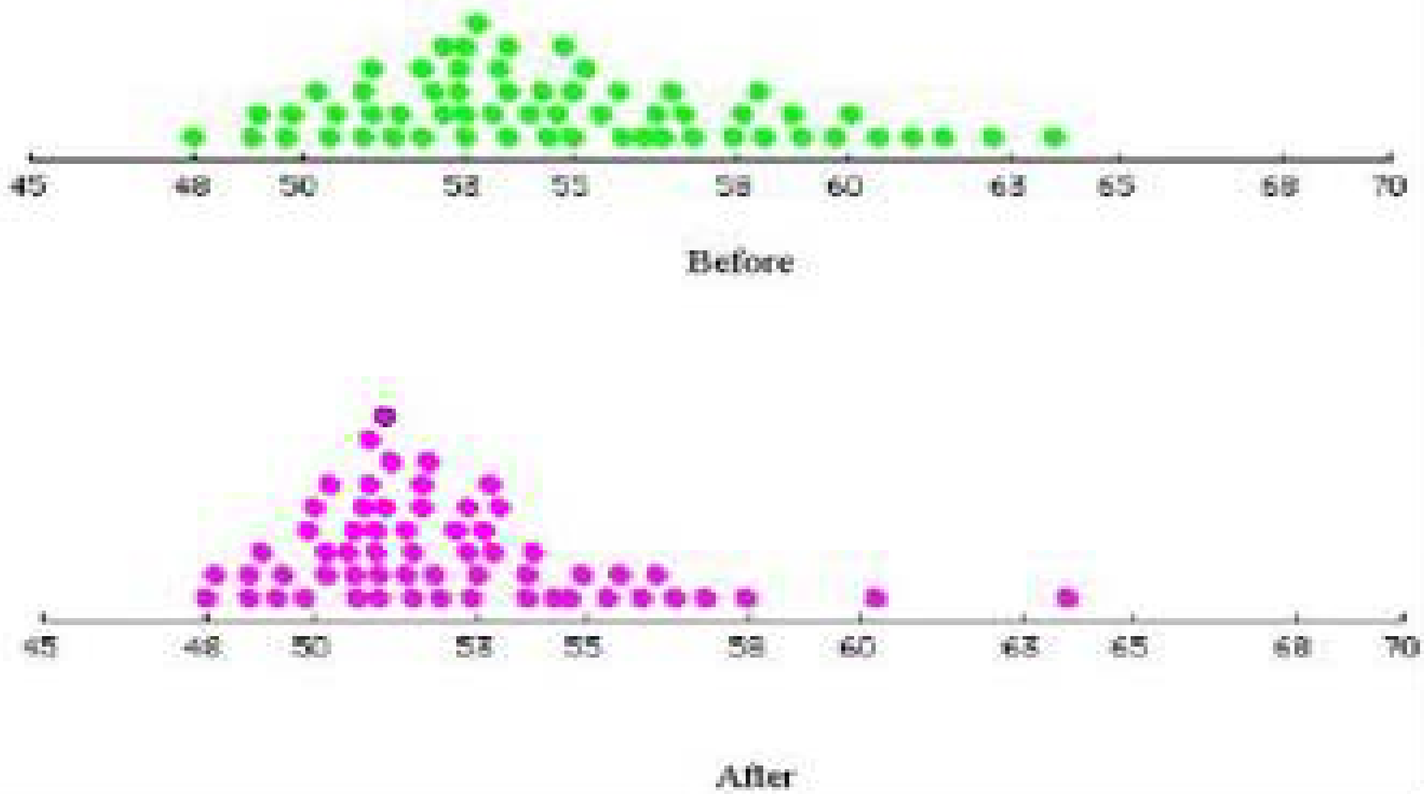
Minitool 2



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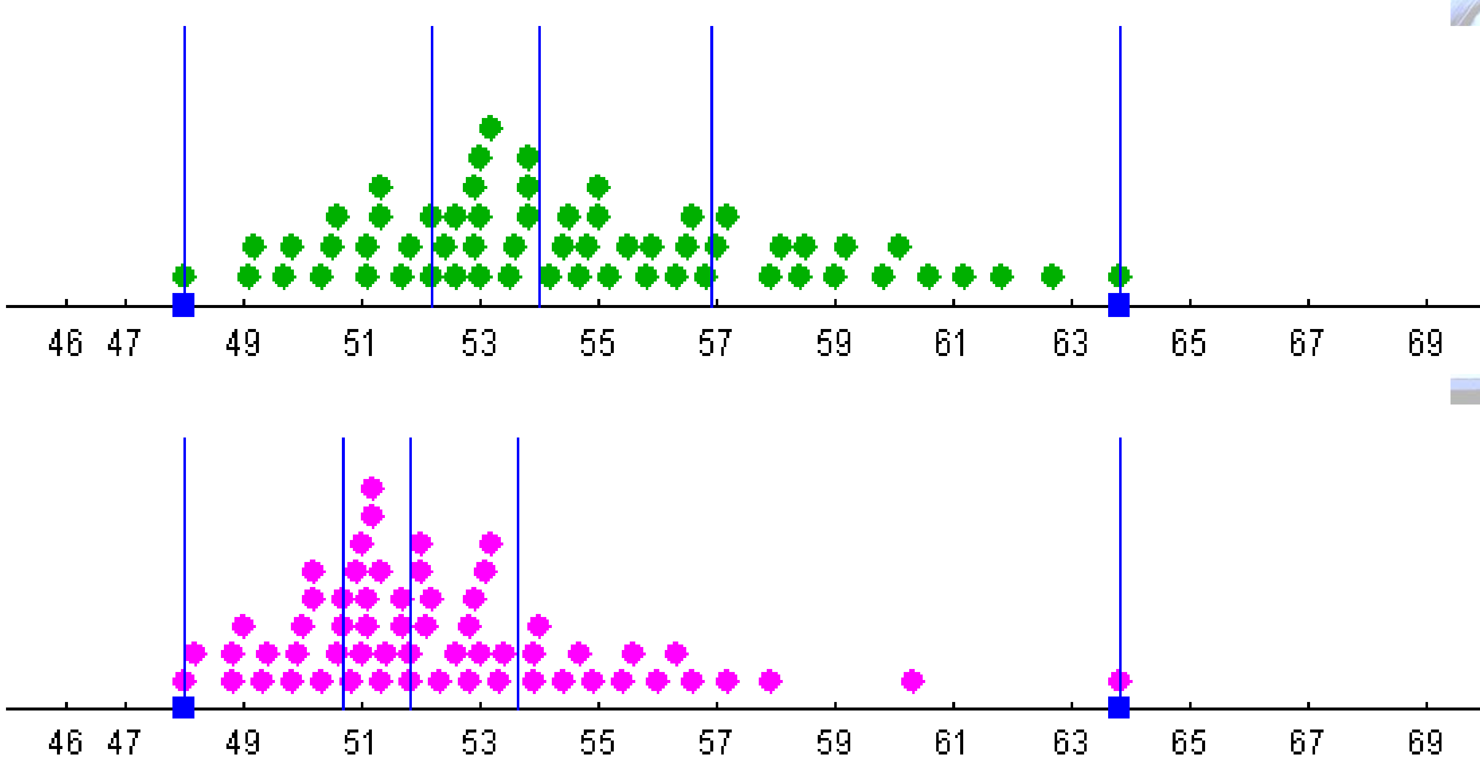
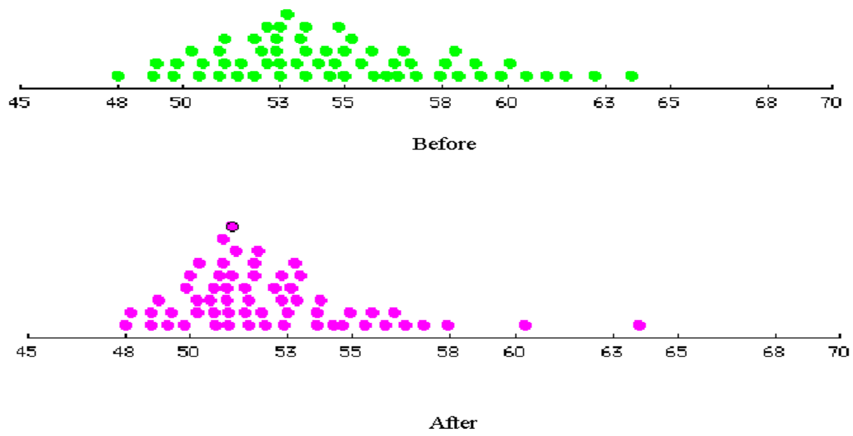
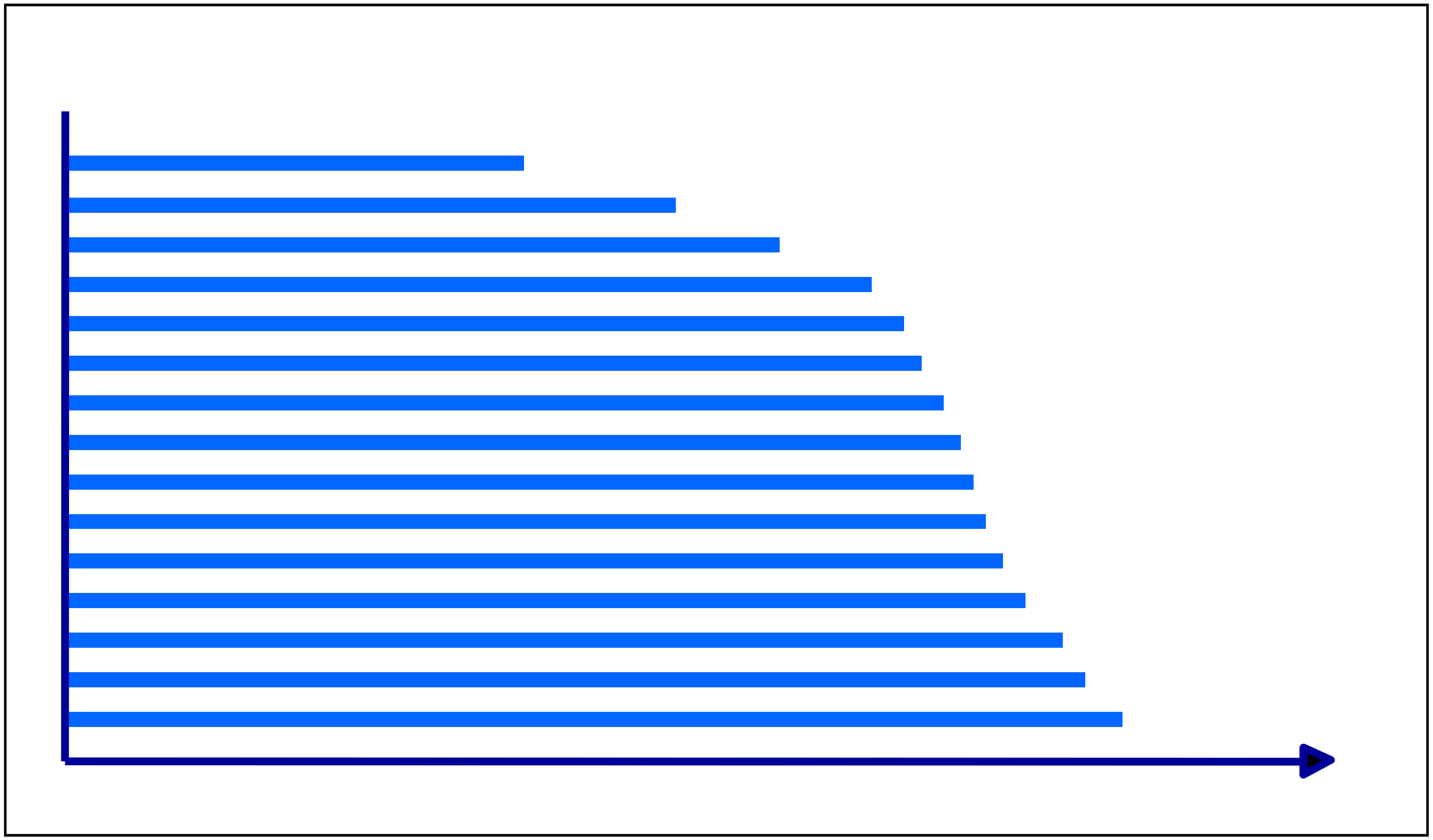
Speedtrap

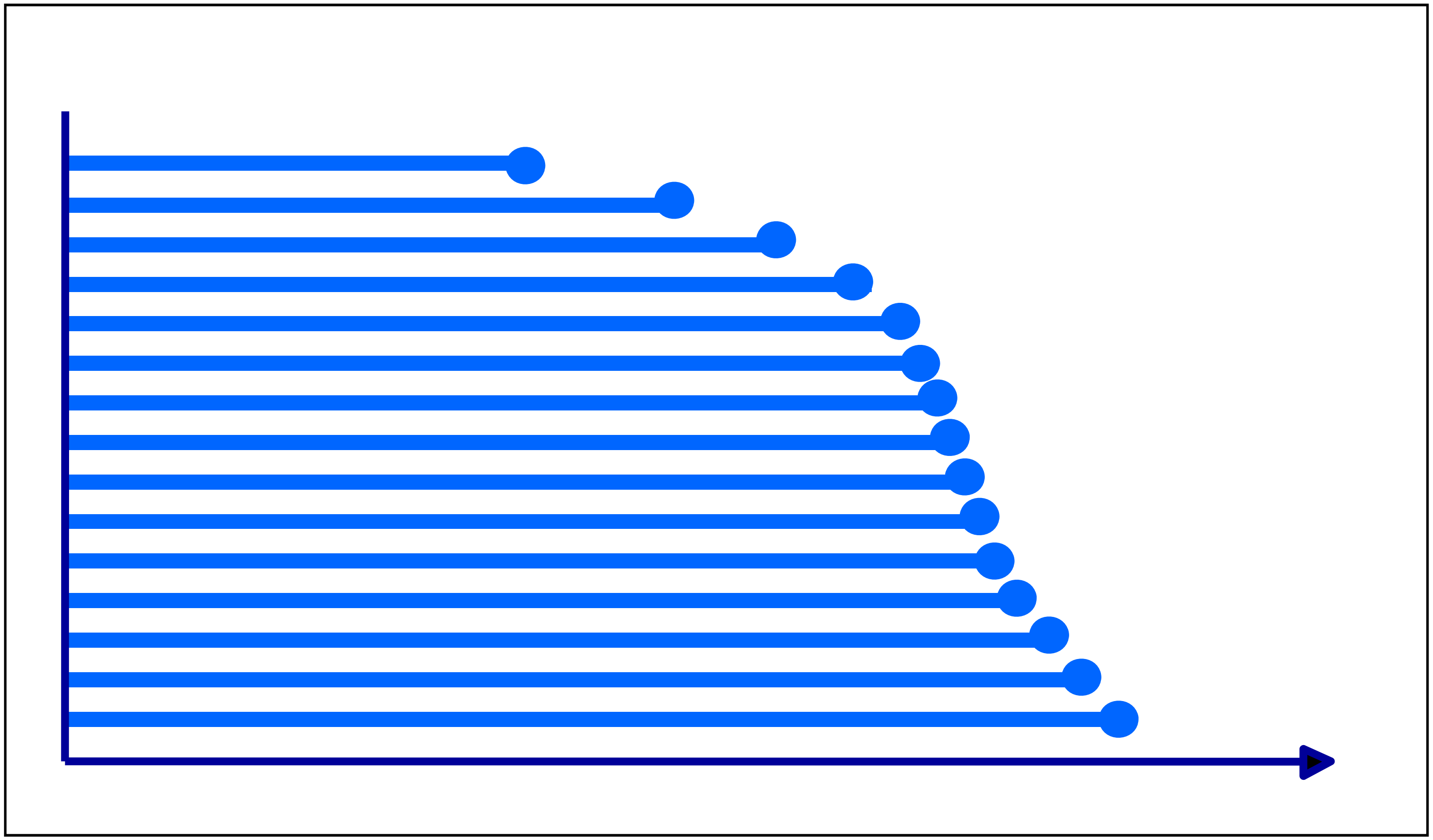


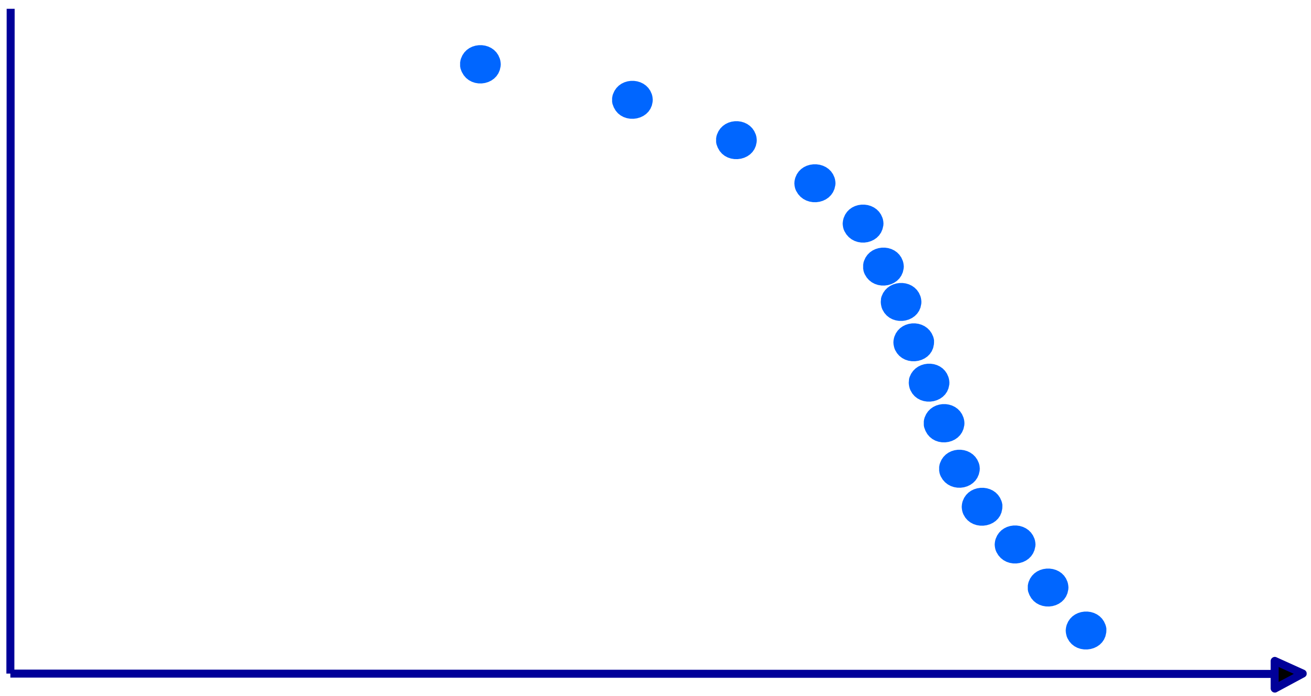


- Janice: If you look at the graphs and look at them like hills, then for the before group the speeds are spread out and more than 55, and if you look at the after graph, then more people are bunched up close to the speed limit which means that the majority of the people slowed down close to the speed limit.

Sub-models as the backbone of the instructional sequence







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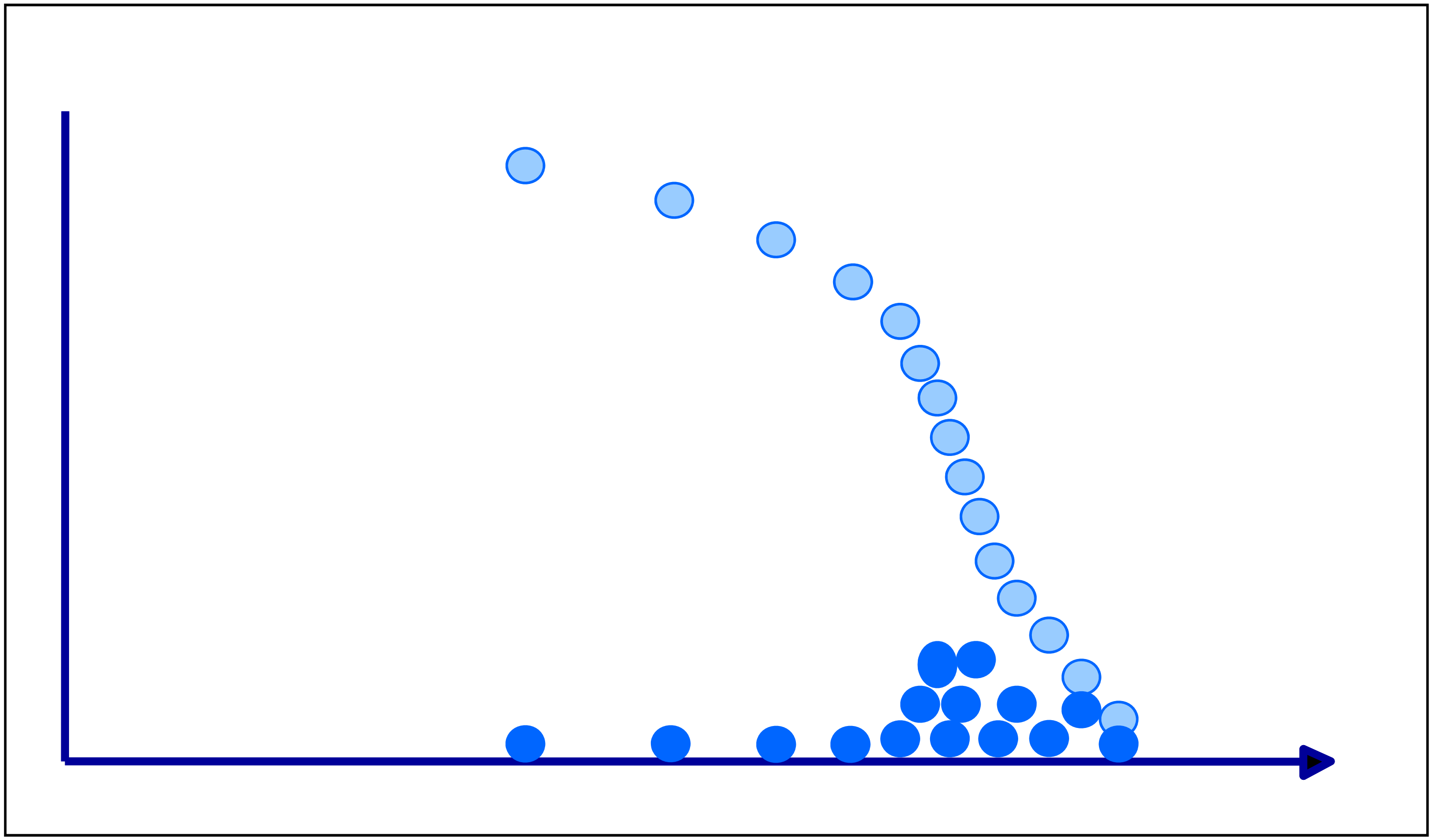

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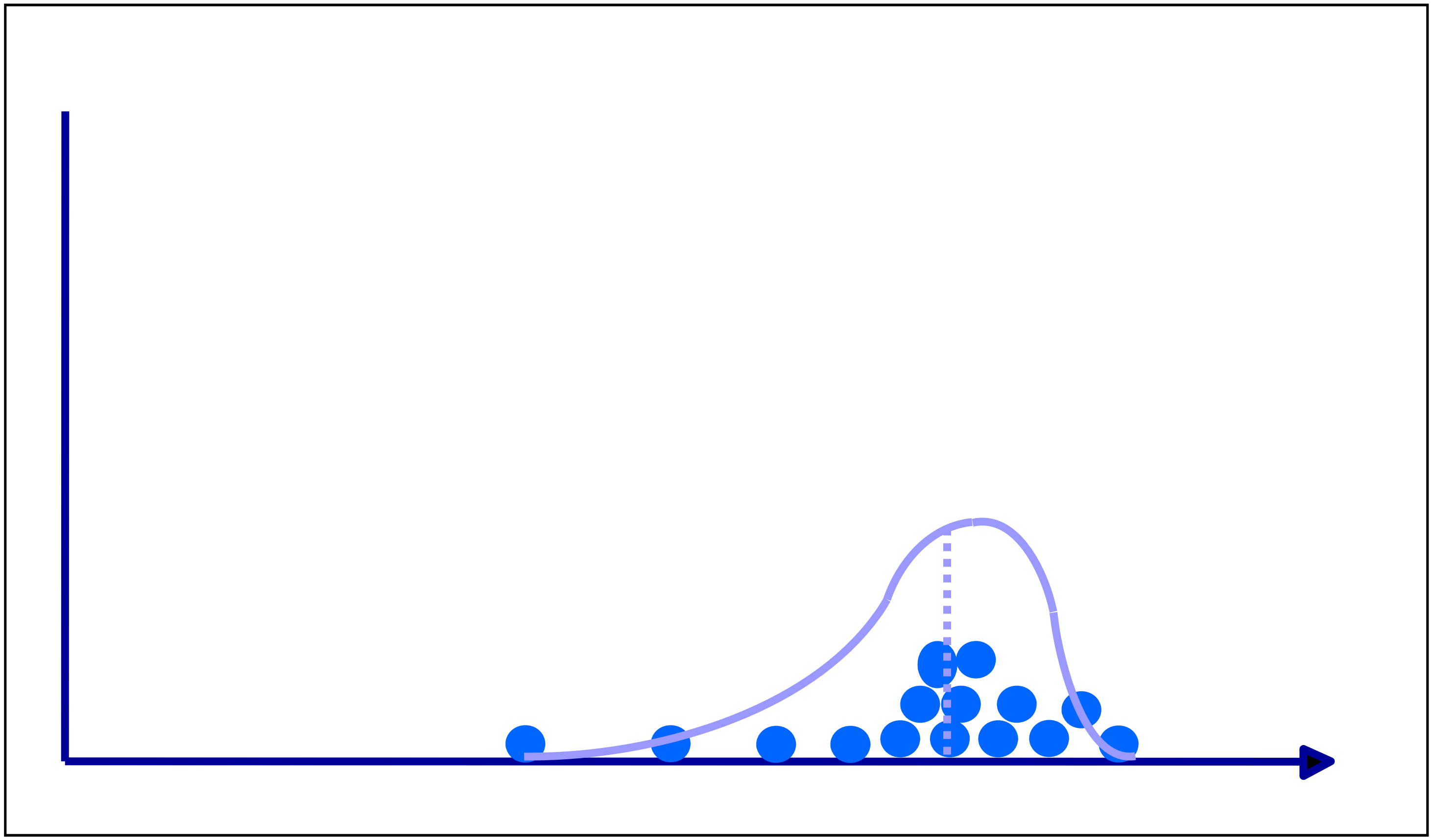

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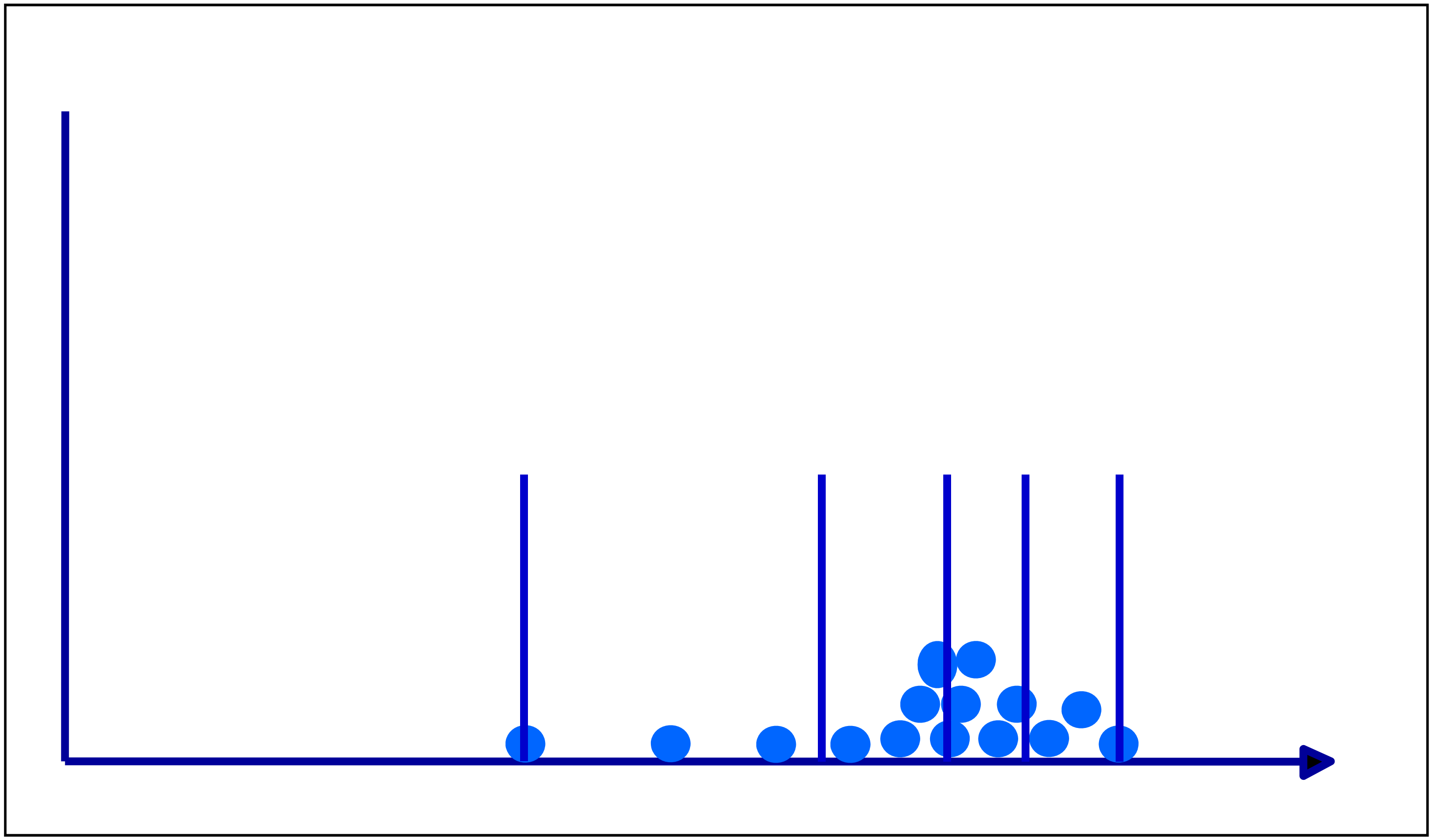

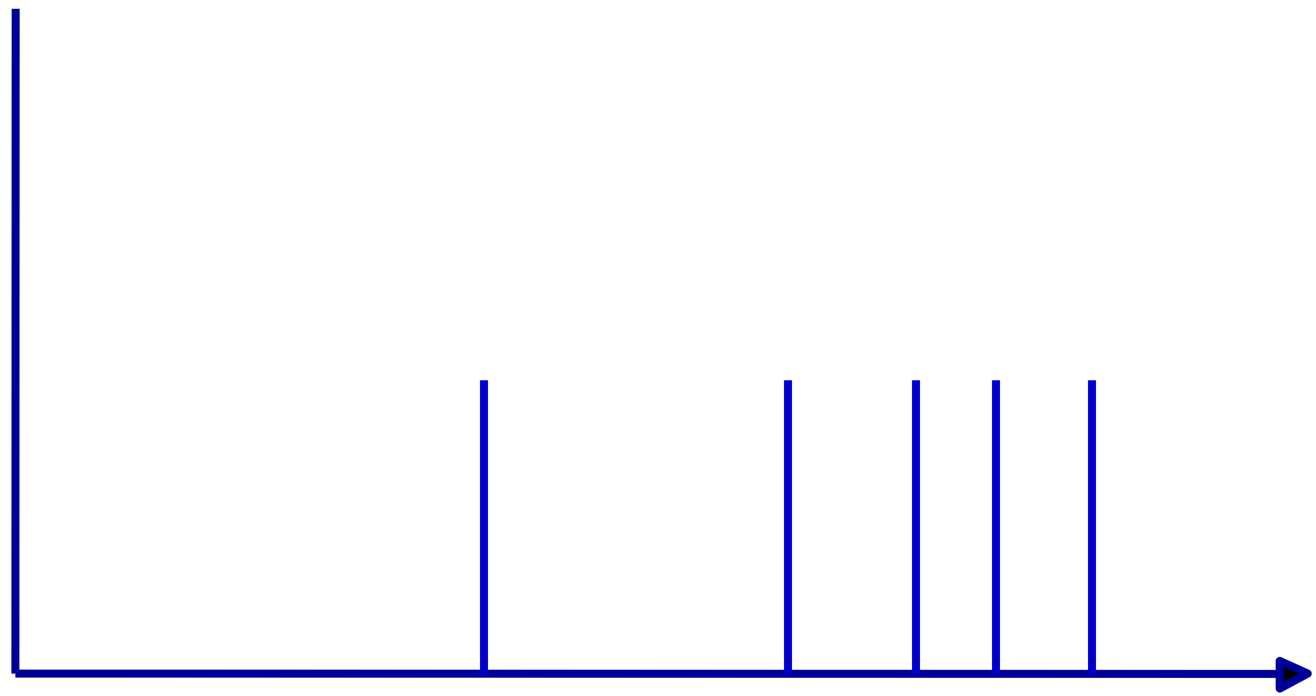

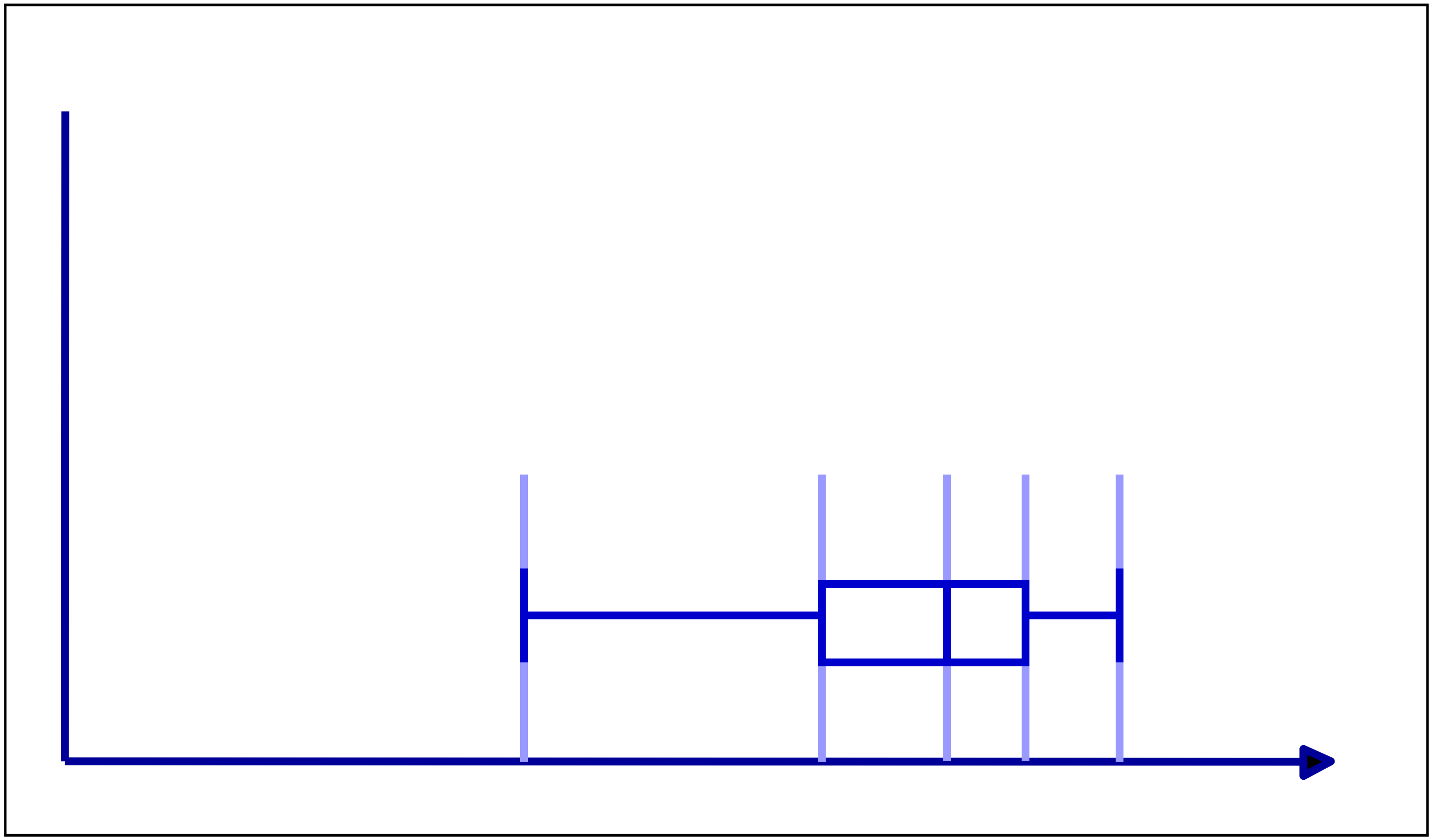

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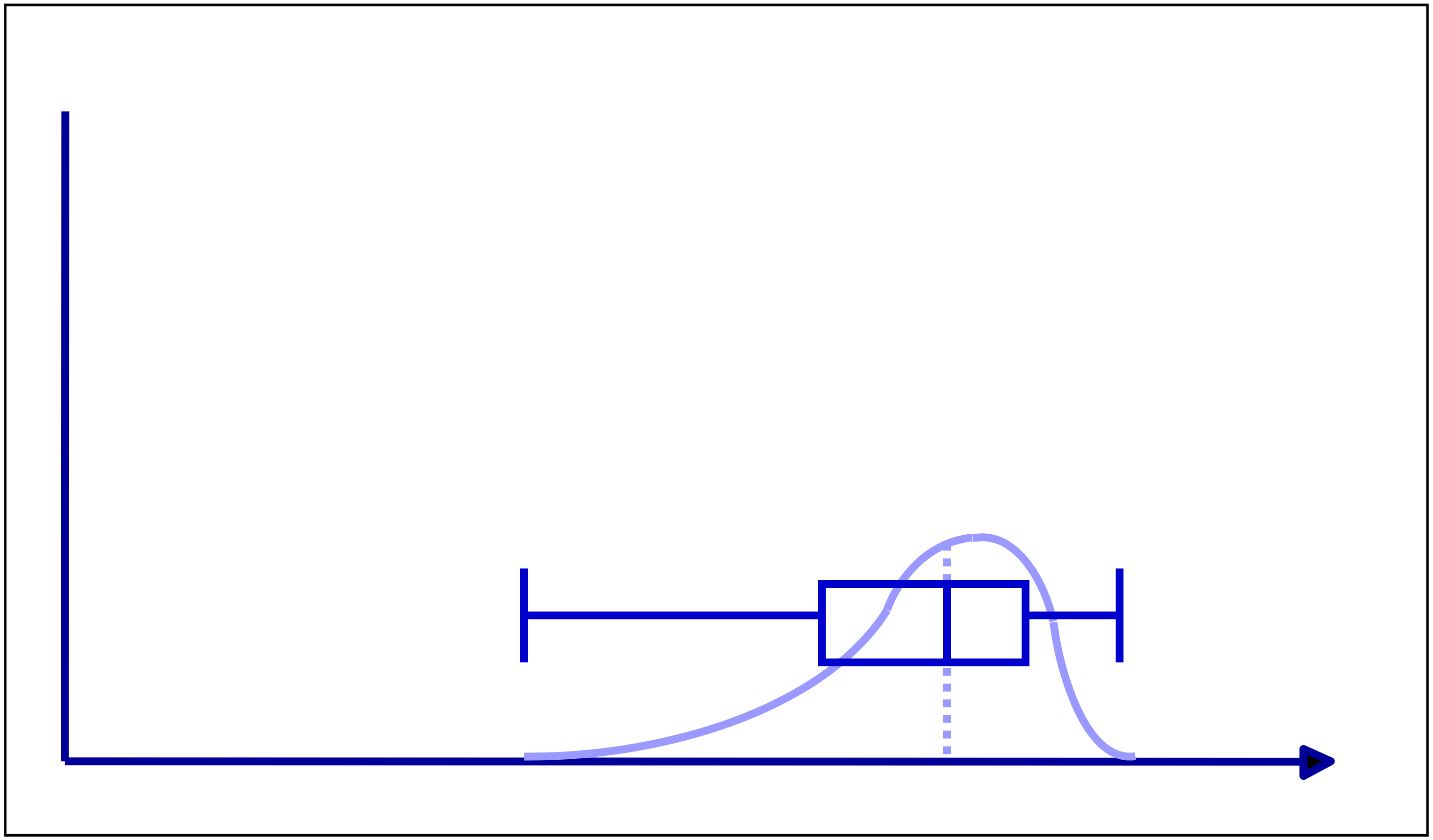

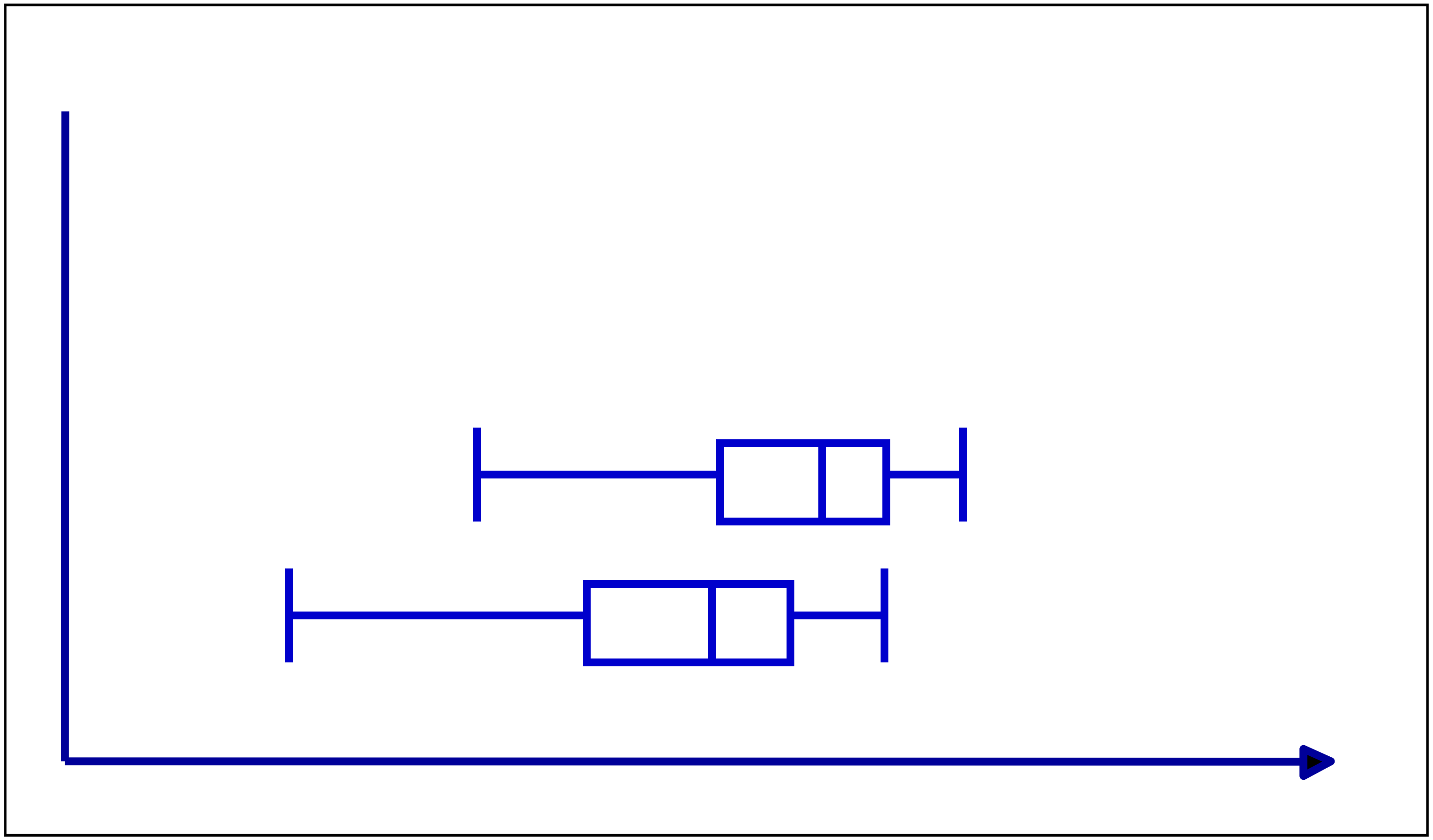

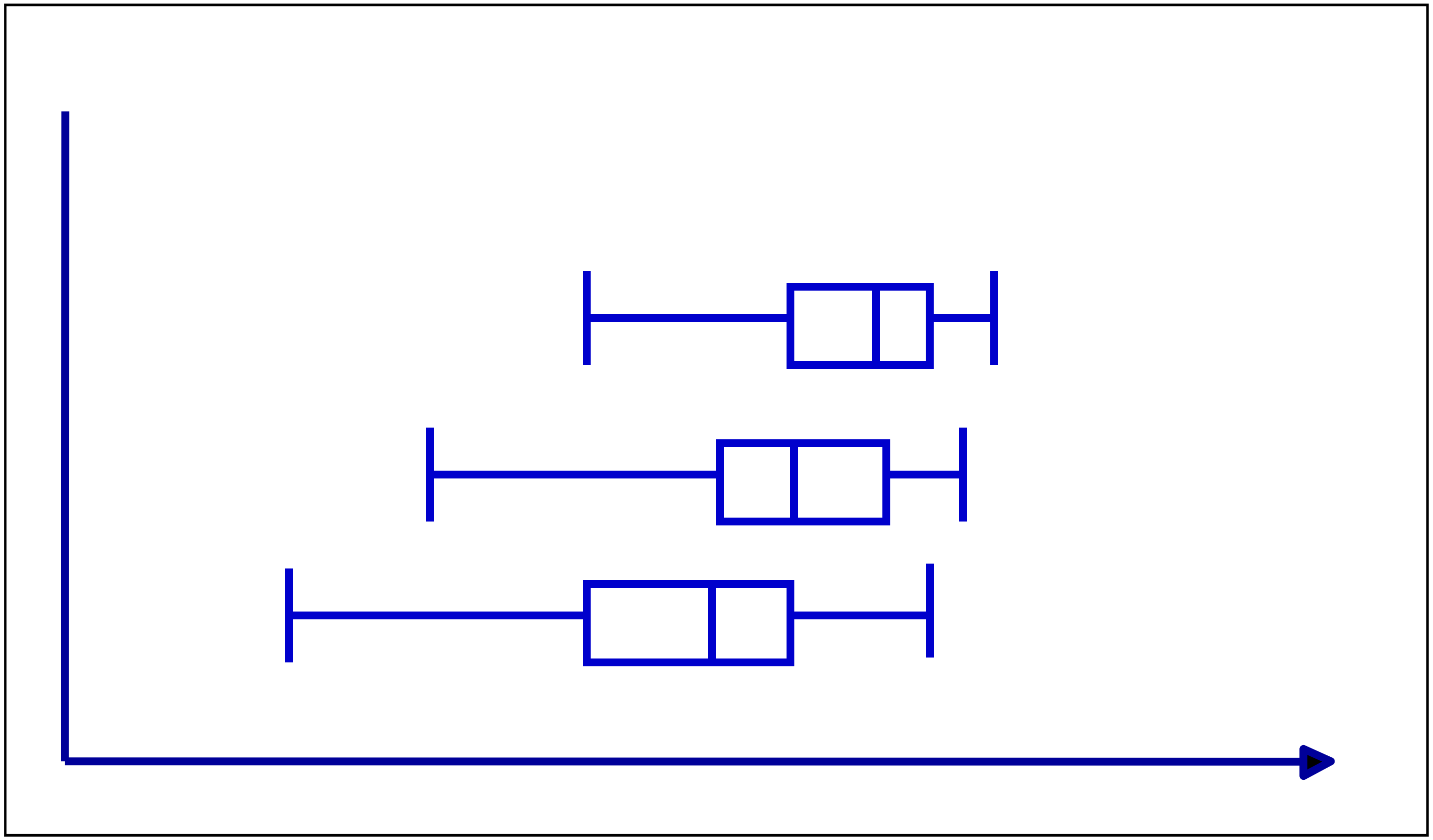

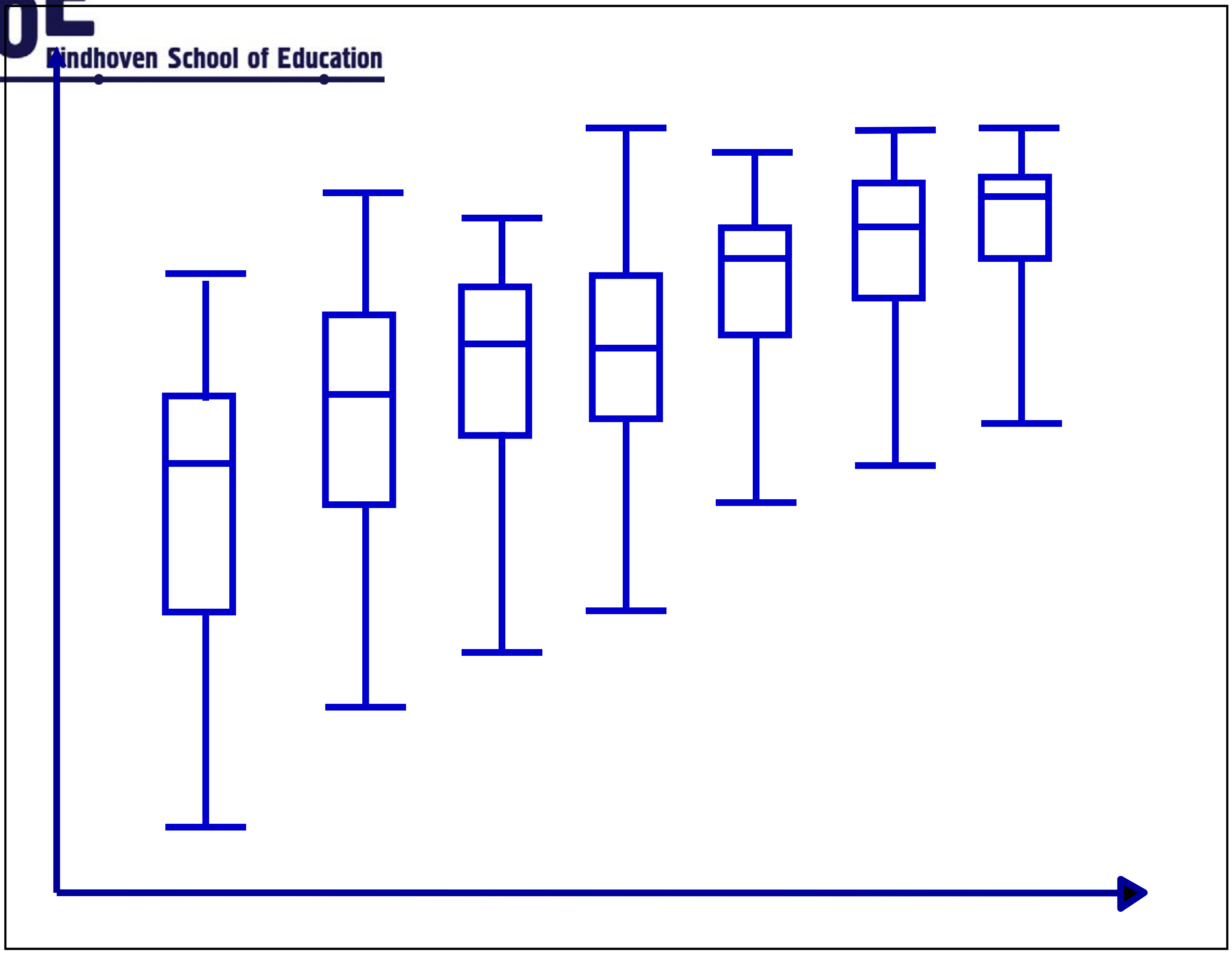

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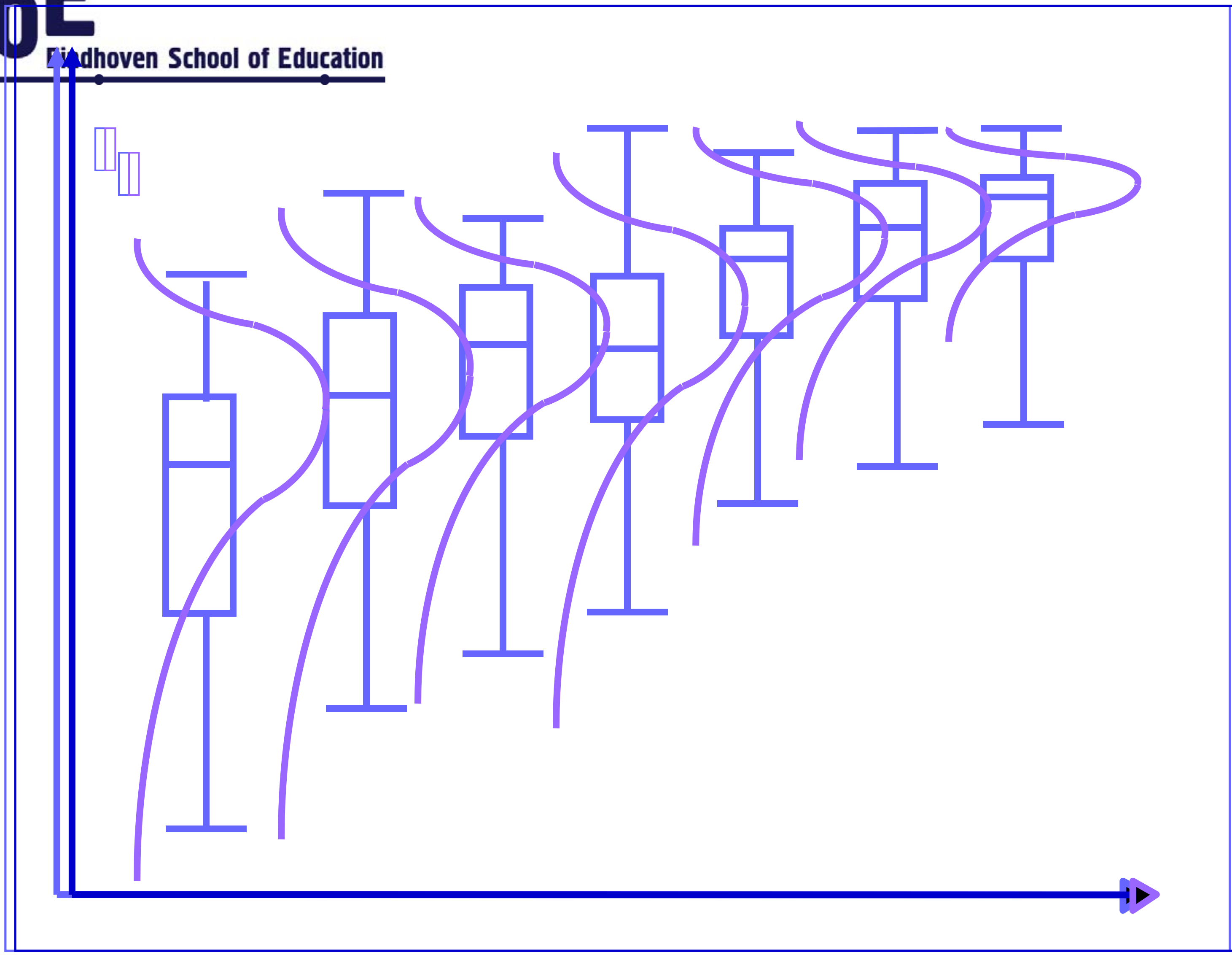
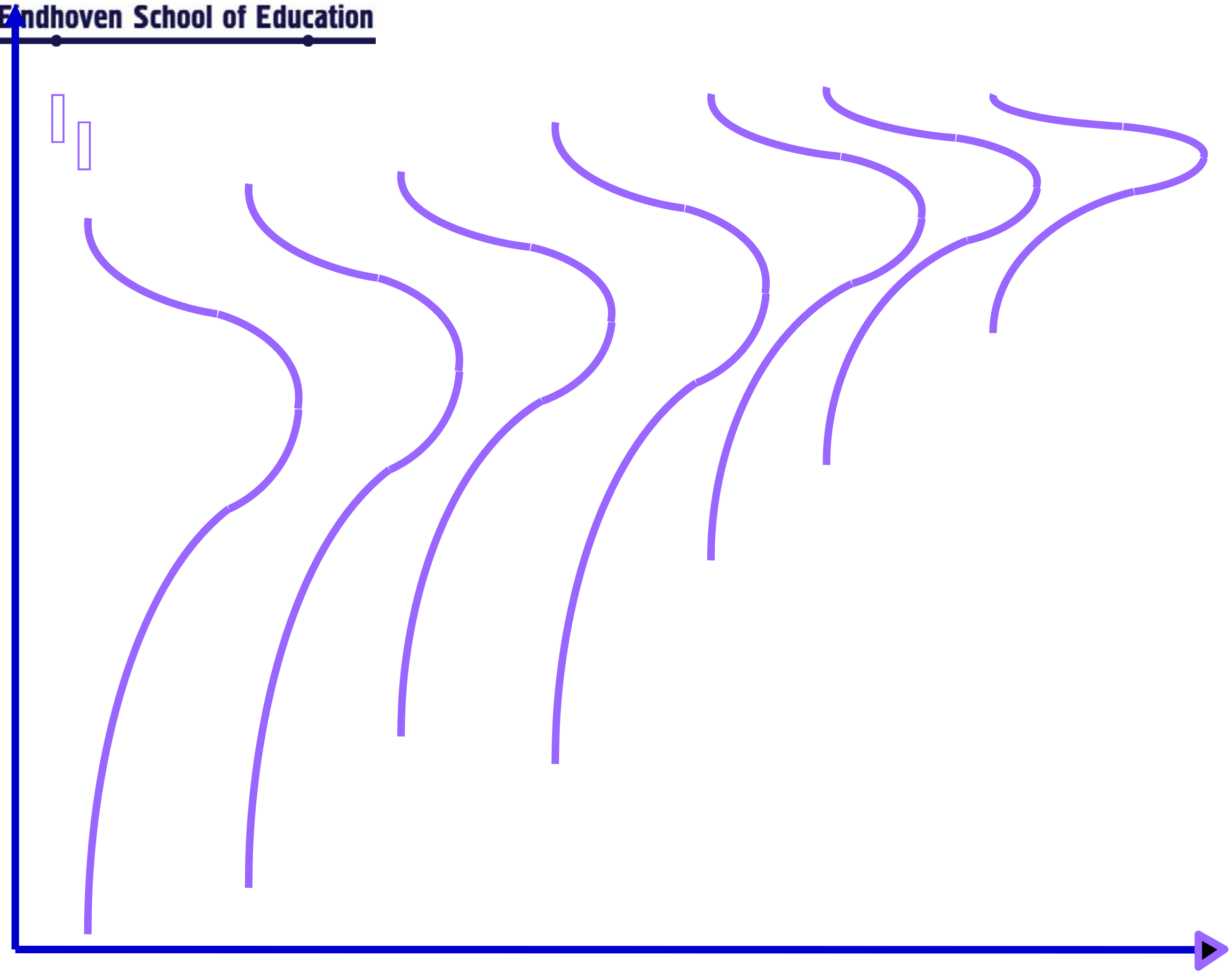
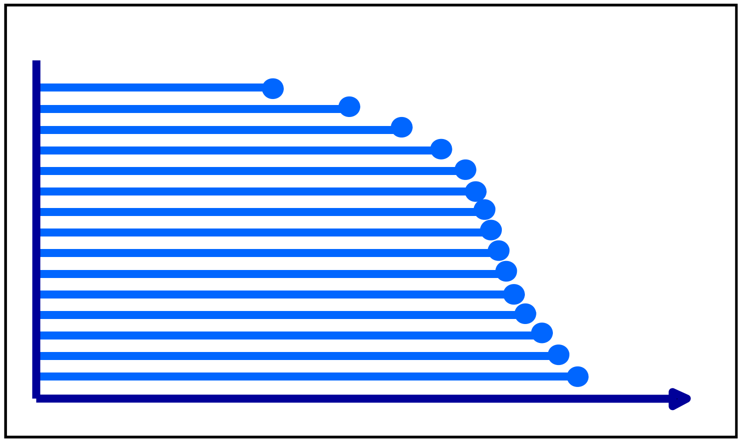




Emergent models

- *Model of* a set of individual measures



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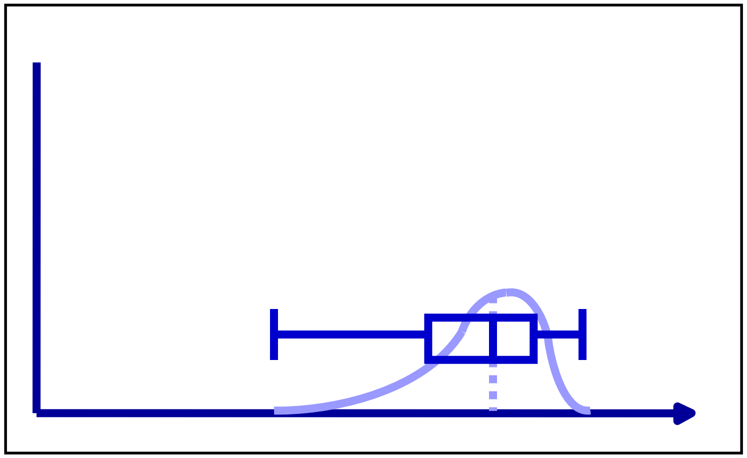
Emergent models

- *Model for* reasoning about a distribution



  - skewed to the left
  - majority/density

Emergent Modeling

Situational level:

Activity in the task setting, in which interpretations and solutions depend on understanding of how to act in the setting (often out of school settings)

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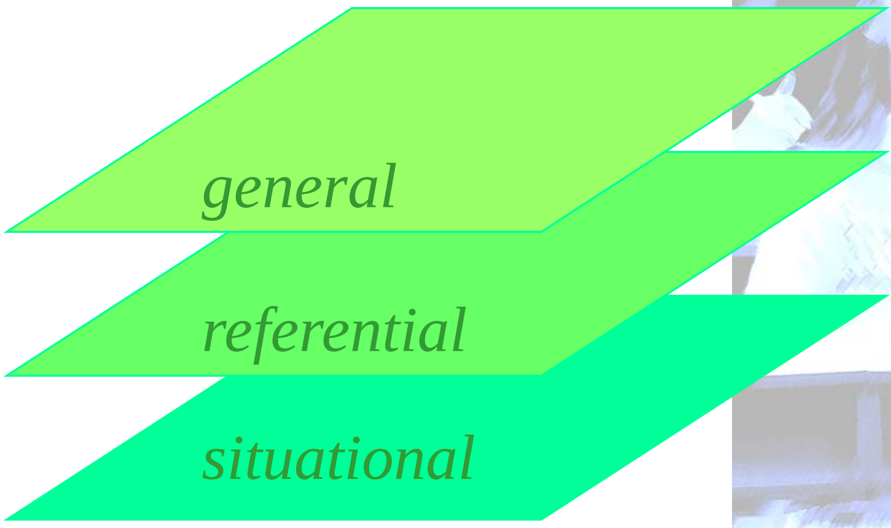
Emergent Modeling

Referential level:
Referential activity, in which the model derives its meaning from the reference to activity in the task setting, and functions as a <u>model of</u> that activity.

Emergent Modeling

General level:

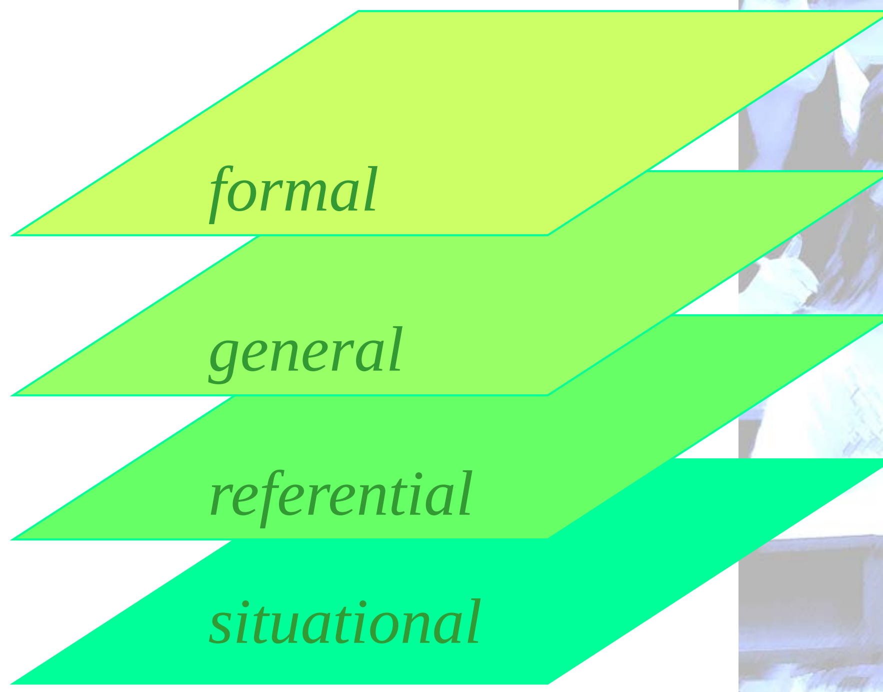
General activity, attention shifts towards mathematicals relations, the model starts to derive its meaning from those mathematical relations, and becomes a <u>model-for</u> more formal mathematical reasoning



Emergent Modeling

Formal level:

Formal mathematical reasoning, which is no longer dependent on the support of models



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model of => model for

- A *model of* informal mathematical activity becomes a *model for* more formal mathematical reasoning
- Imagery: new symbolizations signify earlier activities with earlier symbolizations for the students

How to help students invent what you want them to invent?

Students construct or invent their own knowledge

> How to help students invent what you want them to invent?

A. Socratic lesson

B. Hypothetical learning trajectory

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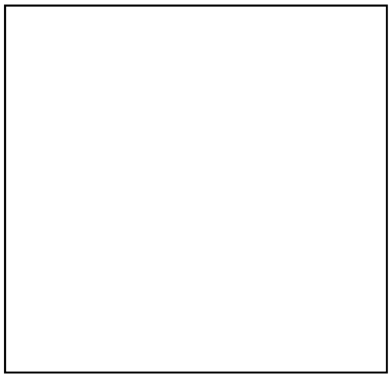
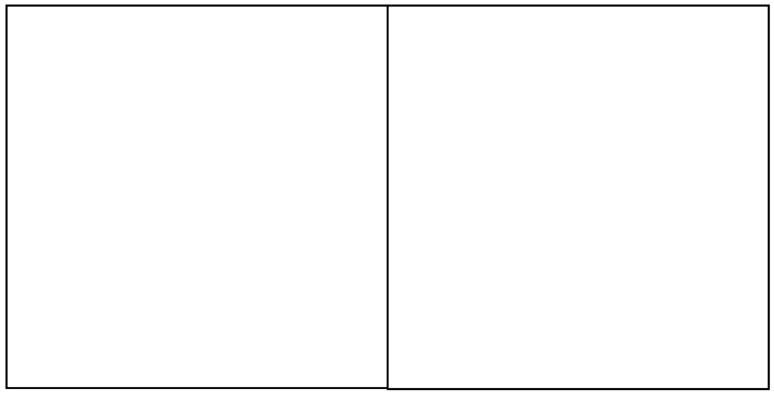
A. Socratic Lesson

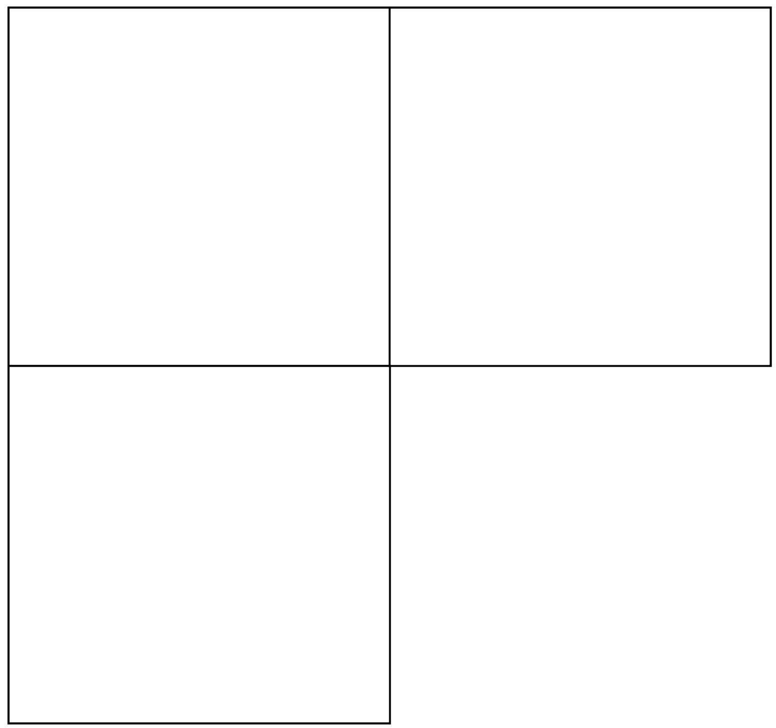


Tell me, is not that our square of four feet? Don’t you agree?
Yes, indeed



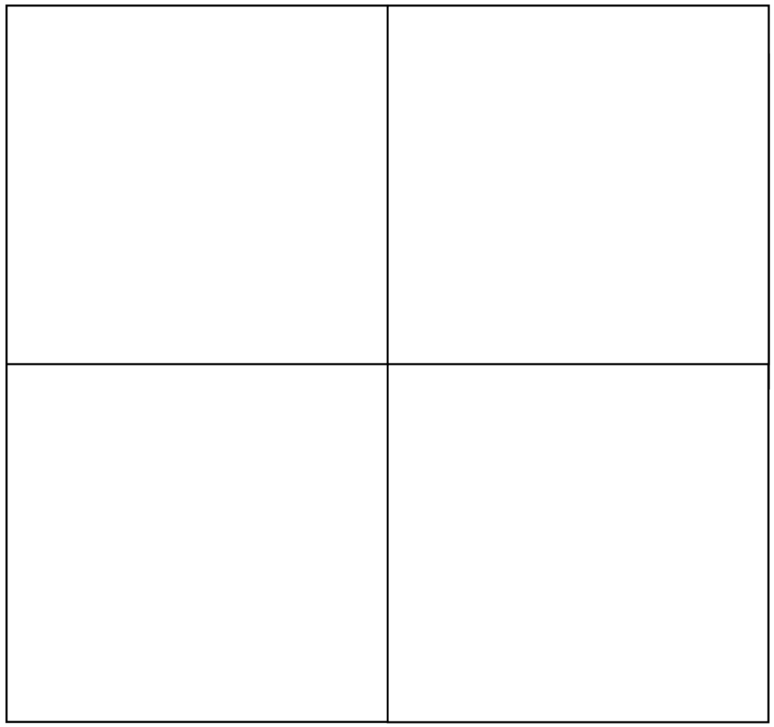
Can we add a similar square here?

Yes.



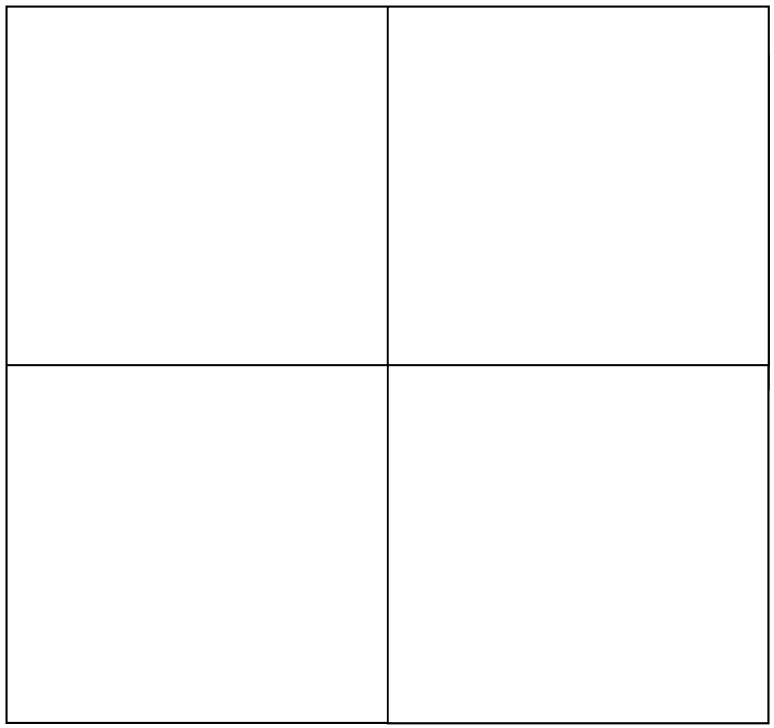
And, cannot we add a third here?

Yes.

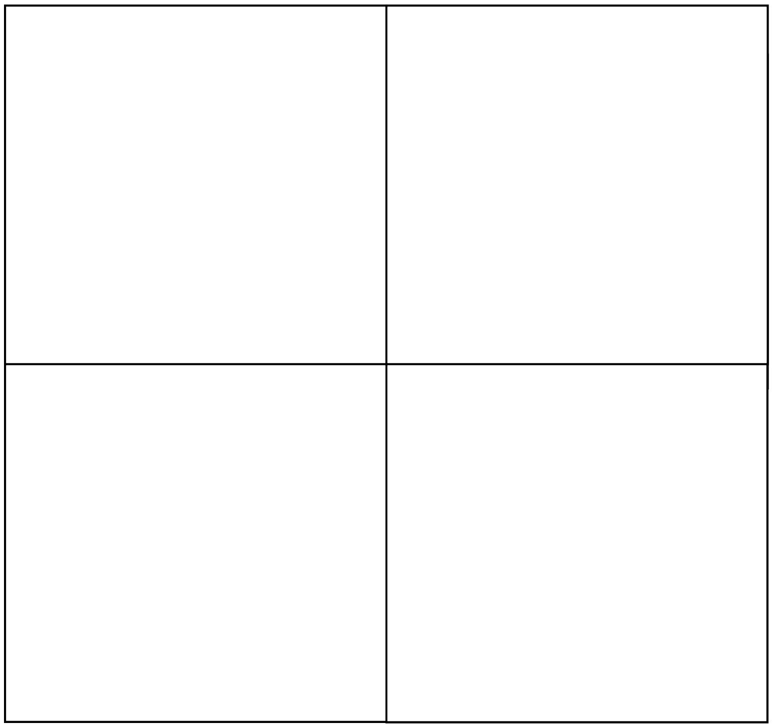


And cannot we fill this corner with a fourth square?

Yes.

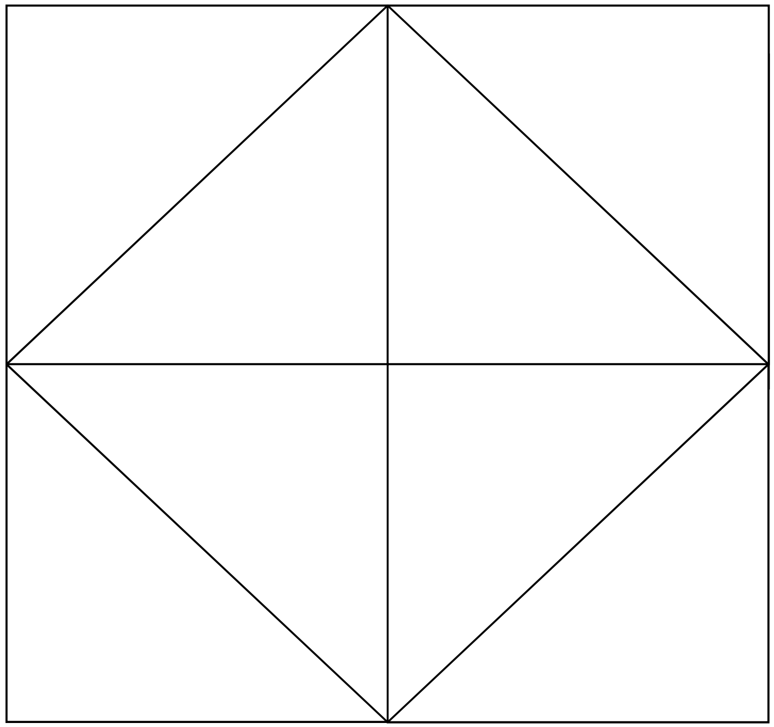


- So, how many times as big is all this compared with our initial square?
- Four times as big.



But we were looking for a square twice as big, remember?

Yes, that is correct.



Does not such a line that goes from this vertex to that vertex, divide each square into two equal parts?

Yes.

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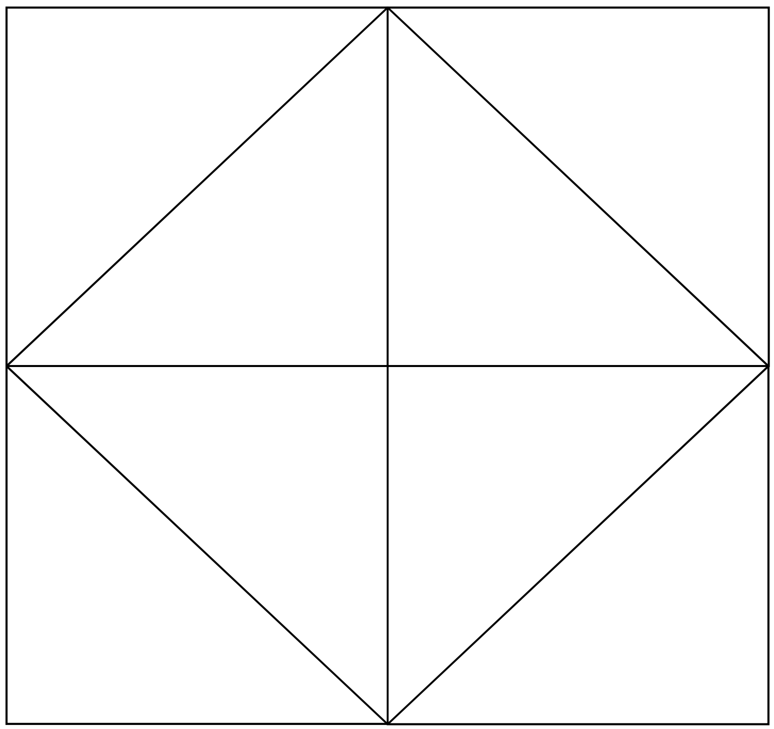
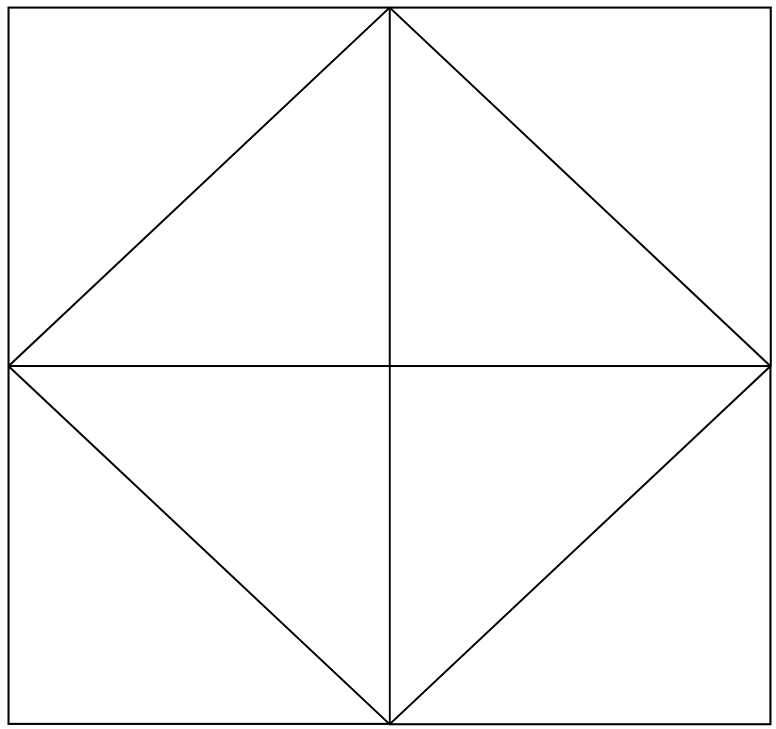
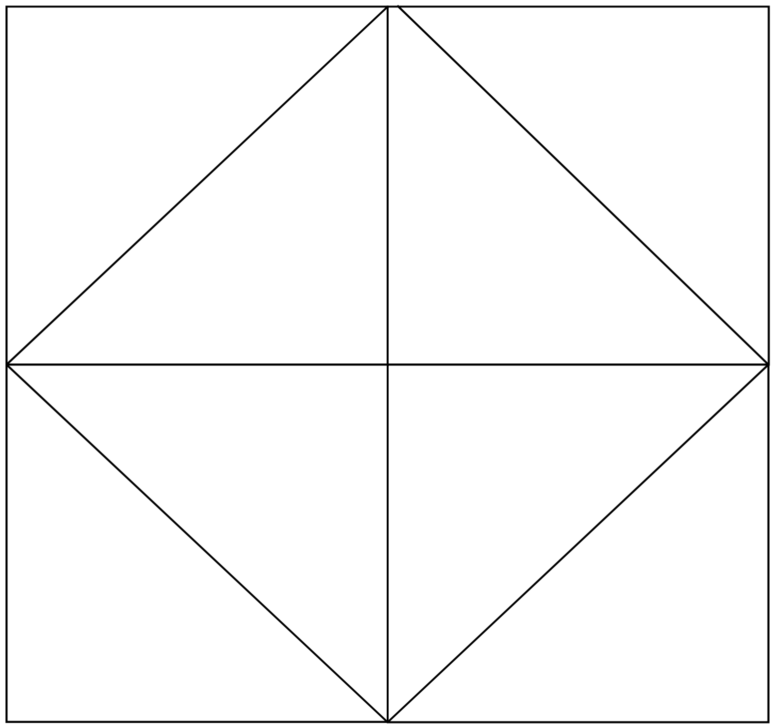


So we get four identical lines that together form a square? Yes.
No, think about it. How big is that square?
I do not know how to figure that out.



Are not these four squares, and does not each line cuts away half of each square? Yes.
So, how many half squares are there? Four.



And how many are there in the original square? Two.
And what is the ratio of four compared to two? It is its double.

> Who does the thinking?

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Helping students to reinvent

B. Hypothetical learning trajectory

- Different role for the teacher: Choosing tasks with an eye on what it might bring about ⇔
  - envision the mental activities of the students
  - anticipate how their thinking might help them to develop mathematical insights = “hypothetical learning trajectory” (Simon, 1995)

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HLT, Area as an example (Simon)

Discussion of “Area” with student-teachers

- Area = length x width ?
- Blind algorithm??

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Rectangles problem 1.

Determine how many rectangles, of size and shape of the rectangle that you were given, could fit on the top surface of your table. Rectangles cannot be overlapped, cannot be cut, nor can they overlap the edges of the table. Be prepared to describe to the class how you solved this problem.

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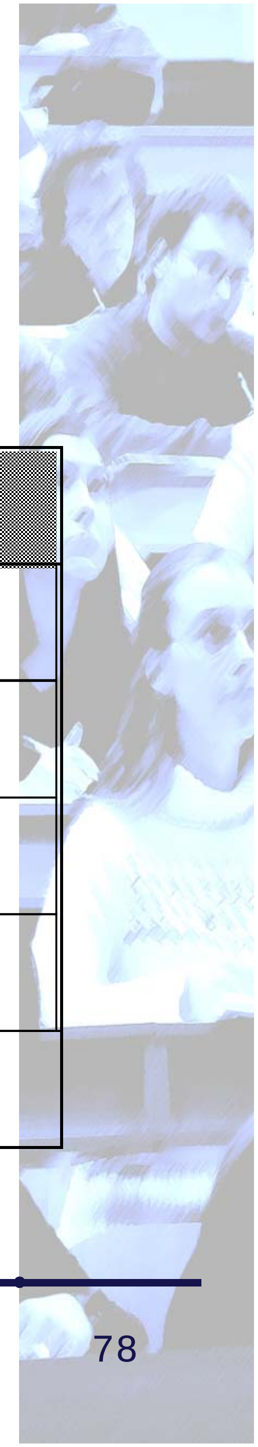
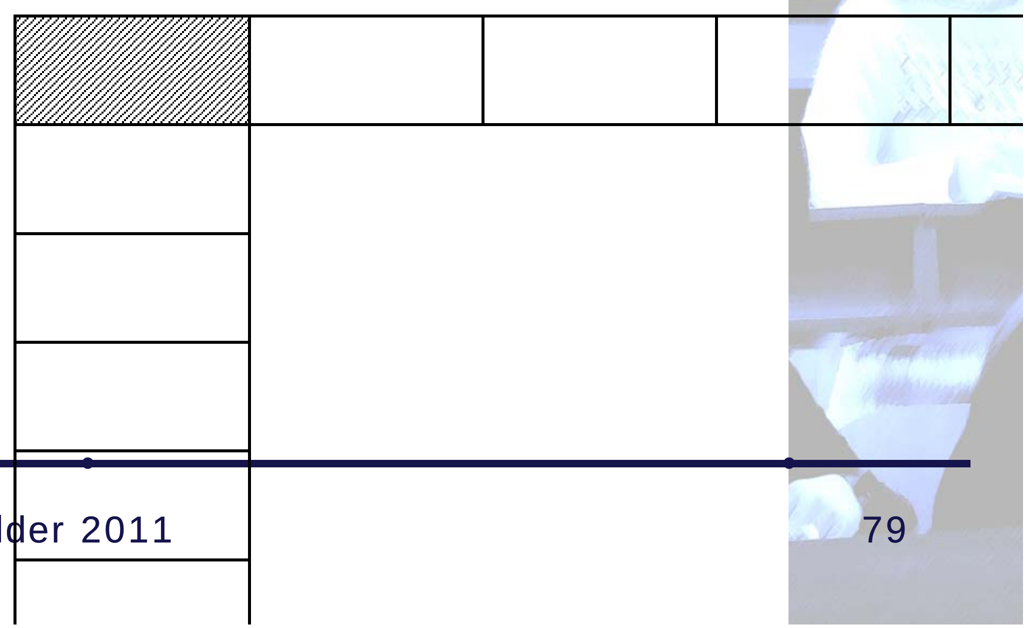


Rectangles problem 2.

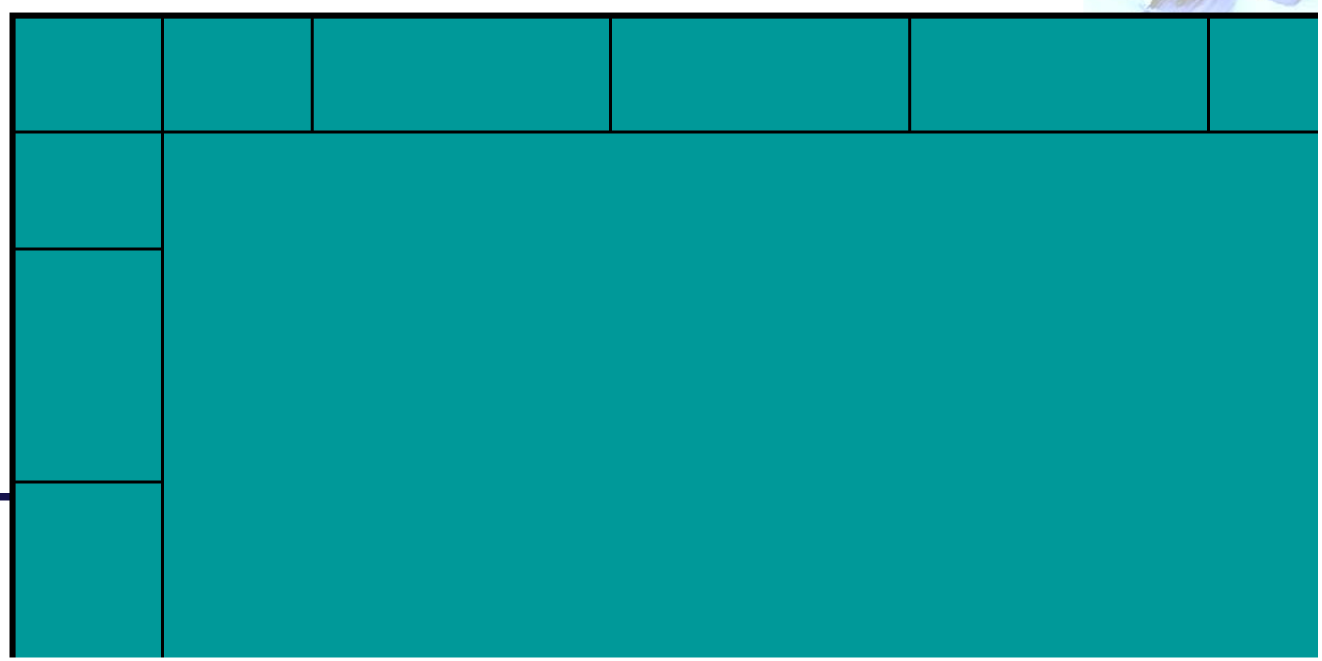
Bill said, “If the table is 13 rectangles long and 9 rectangles wide, and if I count 1, 2, 3, ..., 9, and then I multiply, 13 x 9, then I have counted the corner rectangle twice.” Respond to Bill’s comment.



Rectangles problem 3.

I used the turned rectangles method, and I got 32 for table A, and 33 for table B.

Can we now say something about which table is bigger?



The stick problem.

Two people work together to measure the size of a rectangular table; one measures the length and the other the width. They use a stick to measure with. The sticks, however, are of different lengths.

Louisa says, “The length is four of my sticks.”

Ruiz says, “The width is three of my sticks.”

What can you say about the area of the rectangular table?

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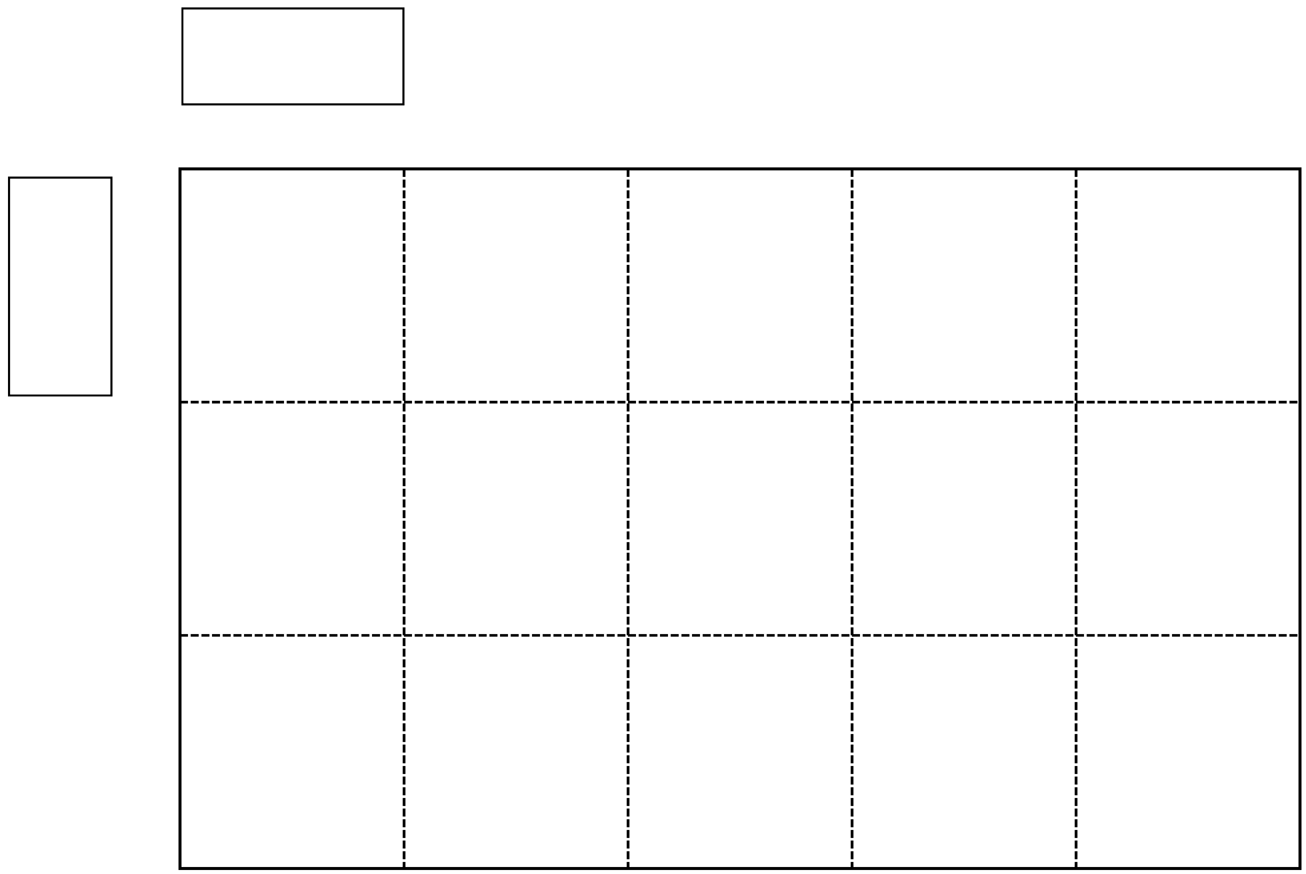
Original turned rectangles problem



Helping students to invent

- The hypothetical learning trajectory is very different from the classic Socratic way of developing an idea in interaction with the students:

  > Who does the thinking?

Helping students to invent

- Doubling square (2), in the Socratic lesson, the teacher does the thinking; teacher checks whether the students follows along;
- Area Simon (1), with the hypothetical learning trajectory, the students do the thinking; the teacher tries to adapt to what the students are thinking.

Helping students to invent

- Anticipate in advance what the mental activities of the students will be when they will participate in some envisioned instructional activities
- Try to find out to what extend the actual thinking processes of the students correspond with the hypothesized ones
- Reconsider potential or revised follow-up activities

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How to ensure student participation in problem oriented mathematics?

How to ensure student participation in problem oriented mathematics?

- Problem oriented mathematics: students have to engage in a problem solving activity
- "Resistance" of students to teachers' attempts to implement a problem solving approach (Desforges & Cockburn; Hiebert & Stigler):
- Explanation:
  - That is what they are used to: reproducing the teachers reasoning and procedures (Didactical contract)
- Communication ⇔ expectations: An everyday-life example

Communication ⇔ expectations

- Elicitation pattern
  - Could you please tell me the way to West Street?

Communication ⇔ expectations

- Elicitation pattern
  - Could you please tell me the way to West Street?
  - You take the second on the right, then the first on the left, and then you walk straight into West Street.

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Communication ⇔ expectations

- Elicitation pattern
  - Could you please tell me the way to West Street?
  - You take the second on the right, then the first on the left, and then you walk straight into West Street.
  - OK, the second on the right, then the first on the left, and then straight ahead. Perfect, well done.

    Now, could you also tell me the way to Central Avenue?

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Social norms regular classrooms

- Based on the teacher's and the textbook's authority
- Social Norms
  - Try to find out what the teacher has in mind
  - Follow the teacher's solution methods
  - Ask questions if you don't understand
  - The teacher has the obligation to explain

Social norms problem oriented classrooms

- Based on the intellectual autonomy of the students and the idea of a Learning community:
  - Learning as a group: learning from and with each other
- Social Norms
  - Obligations to explain & justify; try to understand, ask for clarification & challenge

How to change the classroom culture?

- Establishing social norms ⇔ experience
  - What is valued
  - What is rewarded
- Using instances as opportunities to clarify norms
- Example: Donna (taken from Erna Yackel)

- Mr. K.: “How many?”
- Donna: “Eight”
- Mr. K.: “How many?”
- Donna: “Eh, … seven(?)”

Next Mr. K. moves to other students. Later as it is Established that 8 was the right answer, Donna complains

- Donna: “I said eight but you said I was wrong!”
- Mr. K.: “What is your name?”
- Donna: “Dona”
- Mr. K.: “What is your name?”
- Donna: “Dona”
- Mr. K.: “And if I would ask you again, “What is your name?”, would you say anything else but Donna?”

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Cultivating an inquiry classroom culture

- Asking for explanations
  - Please explain your answer. (Why is that so? How do you know?)
- Asking for clarifying questions
  - Who has a question for Jim?
- Pass the problem along
  - Who can answer Paula's question?
- Asking for a personal judgment
  - Ann says that it will cost $16.25, do you agree?
- Promoting that students listen and try to understand
  - Did you follow what he said, could you explain it to me?

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Cultivating an inquiry classroom culture

- Revoicing (for instance to help students to follow the argument)
- Modeling favorable behavior
- Showing genuine interest in the student’s thinking
- Building on the input of the students

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- Mathematics in the City (CUNY, Cathy Fosnot, Maarten Dolk et al.)

- Thanksgiving
- The turkey weighs 24 pound, and it takes 15 minutes per pound
- How long does it have to cook?

Turkey video, Cultivating an inquiry classroom culture

- "Try to explain in such a manner that everybody can understand."
- "Listen carefully, and see if you understand."
- "Who thinks he can explain what Amber and Vicky tried to do?"
- "Do you have something to add?"
- "Without telling them how many hours could you explain to them how they could figure that out?"
- "Great question, did you understand …"
- "Tell them .."
- "Did you hear what he said ?"

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Foster student participation

- Establishing and cultivating a classroom culture in which students feel obliged to explain, justify, listen, challenge
- Reward and build on student thinking

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How to foster the development of sophisticated mathematics? (Generalizing & formalizing)

Fractions

Algebra

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Fractions

In the Netherlands there is a call for assessment of skills

- However, learning fractions asks for more than correct answers on written tests
- Transition from all sorts of measures: e.g. $\frac{3}{4}$ pizza, $\frac{3}{4}$ kilometer to *rational numbers as mathematical objects*: $\frac{3}{4} + \frac{3}{4} = 1\frac{1}{2}$
- Transition from informal solution procedures to more *formal general rules*

What informal solution procedures might primary school students use to solve the following problem?

- Ailene bought 16 bottles of Coca Cola of ¾ liter.
  > How much liter is that in total?

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Repeated addition

$$\tfrac{3}{4} + \tfrac{3}{4} + \tfrac{3}{4} + \tfrac{3}{4} + \tfrac{3}{4} + \tfrac{3}{4} + \tfrac{3}{4} + \tfrac{3}{4} + \tfrac{3}{4} + \tfrac{3}{4} + \tfrac{3}{4} + \tfrac{3}{4} + \tfrac{3}{4} + \tfrac{3}{4} + \tfrac{3}{4} + \tfrac{3}{4} =$$

> What would be a clever way to carry out this addition?

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What informal solution procedures might primary school students use to solve the following problem?

- Eduardo used three quarter of 16 Gallons of petrol.
  > How much petrol is that?

What informal solution procedures might primary school students use to solve the following problem?

- Eduardo used three quarter of 16 Gallons of petrol.

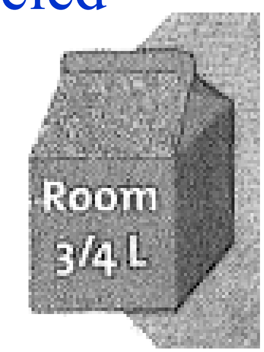
  > How much petrol is that?

  Calculate one fourth --> 16 : 4 = 4

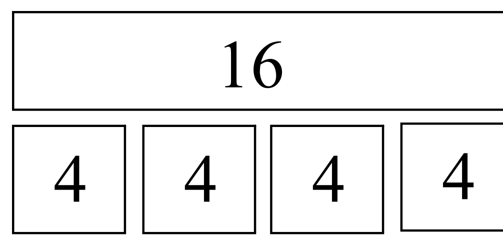
  Thus ¾ is 3 x 4 = 12

Research on Dutch textbooks (Geeke Bruin-Muurling)

- Primary school textbooks aim at number-specific solution methods (end of 6th. Grade)
- number-specific, context related, measures ("labeled numbers")

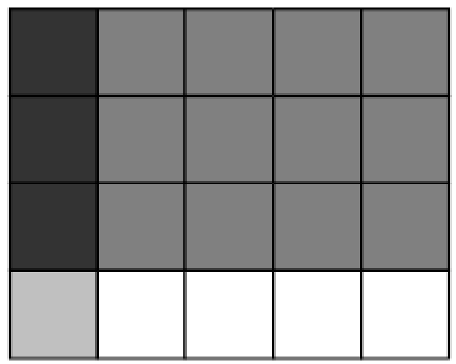


- 16 x ¾ ⇔ 16 cartons of cream of ¾ liter
  - repeated addition ¾ + ¾ + …
- ¾ x 16 ⇔ "¾ part of 16" -->
  - dividing by 4 first 16:4=4; 3x4=12



- Multiplication of fractions <1

  Area model



$$\frac{1}{5} \times \frac{3}{4}$$

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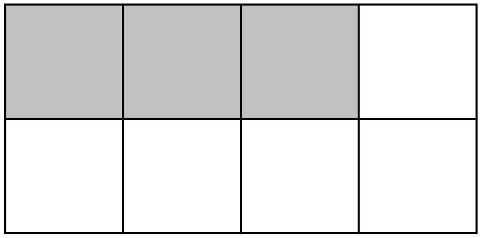
Secondary Education

- The same pictures,
- different meaning

$\frac{1}{2} \times \frac{3}{4}$



$$\text{fraction x fraction} = \frac{\text{nominator x nominator}}{\text{denominator x denominator}}$$

- A general rule for bare numbers (mathematical objects)
- The picture is an illustration/proof for all rational numbers

Numbers from tied to identifyable objects to (mathematical) objects

- Van Hiele levels:
- Ground level: Number tied to countable objects: “four apples”, “four marbles”, “four meters”
- Higher level: 4 is a mathematical object that derives its meaning from a network of number relations: 4 = 2+2 = 3+1 = 5-1 = 8:2
- Mathematical objects in a network of number relations = New mathematical reality

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From fractions tied to identifiable units to rational numbers as mathematical objects

- Fractions tied to identifiable units $\frac{3}{4}$ pizza, $\frac{3}{4}$ kilometer
- Objects = Junctions in a framework of number relations:

  $\frac{3}{4} = \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$

  $\frac{3}{4} = 3 \text{ x } \frac{1}{4}$

  $\frac{3}{4} = 1 - \frac{1}{4}$

  $\frac{3}{4} = \frac{1}{2} + \frac{1}{4}$

  $\frac{3}{4} + \frac{3}{4} = 1\ \frac{1}{2}$

  $\frac{3}{4} = 6/8 = 9/12 = 12/16$

  $\frac{3}{4}$ of 100 is 75 etc.

From number-specific procedures to general rules

- Generalizing & formalizing

  For instance reasoning about why adding ¾ sixteen times (<=> 16 x ¾) gets you the same answer as taking 3 times ¼ of 16 (<=> 3/4 x 16)

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  > Why is that?

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¾ + ¾ + ¾ + ¾ + ¾ + ¾ + ¾ + ¾ + ¾ + ¾ + ¾ + ¾ + ¾ + ¾ + ¾ + ¾ =

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From number-specific procedures to general rules

- Generalizing & formalizing

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  > Why is that?

¾ + ¾ + ¾ + ¾ + ¾ + ¾ + ¾ + ¾ + ¾ + ¾ + ¾ + ¾ + ¾ + ….

3 3

→ How many times can I make four ¾? ⇔ 16 : 4 = ?

→ (16 : 4) x 3 <=> ¾ x 16

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From number-specific procedures to general rules

- Generalizing & formalizing

  For instance reasoning about why adding ¾ sixteen times (<=> 16 x ¾) gets you the same answer as taking 3 times ¼ of 16 (<=> 3/4 x 16)
- This kind of reasoning is lacking in Dutch primary and secondary schools

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Towards University Education

- Relations between relations (highest Van Hiele level)
    - $16 \times \frac{3}{4} = 16 \times 3 : 4$, or $16 : 4 \times 3$
    - $\frac{3}{4} \times \frac{4}{3} = 1$
    - from $\frac{6}{2} = 3$ follows that $\frac{6}{3} = 2$

- Eventually students have to see $16 \times \frac{3}{4}$ and $\frac{3}{4} \times 16$ as one thing (a mathematical object in and of itself)
- See the fraction a^2/b^2 as both axa/bxb and as a/b x a/b.
- For instance in $a^2/b^2 - 1 = (a/b)^2 - 1 = (a/b - 1)(a/b + 1)$

Long-term learning strand; generalizing & formalizing, or vertical mathematization

1. Fractions with identifiable units (pizza's, meters, …)
2. Rational numbers as mathematical objects <=> number relations ¾ = ¼ + ¼ + ¼ = 3 x ¼ = 1 – ¼ = ½ + ¼; ..)
3. Relations between operations; multiplication, division, fractions, proportion (¾ <=> 4 : 3 ; 16 x ¾ = 16 x 3 : 4; 4 x ¾ = 3; …)

Algebra

- *Research in the Netherlands (Irene van Stiphout)*
- Conceptual tasks difficult for secondary school students
- Tasks such as:

If $a\sqrt{b} = 1 + 2a\sqrt{1 + b}$, then a = ...

Results Grade 10, 11, and 12 respectively 0%, 1% and 1% correct

- **Solve: $(x - 5)(x + 2)(x - 3) = 0$**

Grade	9	10	11	12	,
2008	4	29	37	47	%
2009	51	40	52	75	%

Algebra

If $a\sqrt{b} = 1 + 2a\sqrt{1 + b}$

then a = …

solution:

Substitute $\sqrt{b} = K$, and $\sqrt{1 + b} = L$

the equation turns into

$$aK = 1 + 2aL$$

=> $$a(K-2L) = 1$$

- Global Substitution Principle (Wenger)

Algebra

Solve: (x − 5)(x + 2)(x − 3) = 0

- A*B*C=0 → A=0, or B=0, C=0

  (x − 5) = 0

  (x + 2) = 0

  (x − 3) = 0

Arcavi: “symbol sense”; acting ⇔ reflecting

Sfard: “dual nature of mathematics: process - Object”

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Summary

- Fractions
  - number-specific solution procedures
  - no higher Van Hiele levels
- Algebra
  - memorized procedures instead of symbol sense
  - Problems in finding solutions to
    - if $a\sqrt{b} = 1 + 2a\sqrt{1 + b}$, then a = …
    - zero points $(x-6)(x-7)(x-8) = 0$

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Reflective note: Generalizing & formalizing does not evolve as one might hope for

- Students do not generalize in the anticipated direction spontaneously; they do not know which characteristics are relevant, and which are irrelevant
- Natural educational processes work counter productive:
  - mastery asks for practicing; students practice with tasks of a given type; students note accidental characteristics

→ Need for a focused attention for comparing strategies and constructing relations → central role for the teacher

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Conclusion

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- **Use of symbols and tools**

  (Emergent Modeling)

  – From models-of to models-for: Actions with new models or tools have to signify earlier activities for the students

  – Shift in focus: focus on mathematical relations

  → new math reality

- **Helping students to reinvent**

  (Hypothetical Learning Trajectory)

  – Anticipate on students' thinking, try to find out how they think, reflect and adjust

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- Foster student participation
  (Classroom Culture)
  - Establish and cultivate social norms problem oriented mathematics (explain, justify, listen, …)
- Aim for sophisticated mathematics
  (generalizing & formalizing)
  – Identify, frame and discuss mathematical issues in order to foster a higher level of mathematical understanding

Thank you